Virtual Math Tutor 3.1 Companion
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Worksheet ***4. Exercises on Addition and Subtraction***

Movie   ***4. Exercises on Addition and Subtraction***

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Document*: *5. Multiplication of Numbers

Summary   ***5. Multiplication of Numbers***

Movie   ***5. Multiplication of Numbers***

Introducing the Operation Times
The Commutative and Associative Properties of Multiplication
The Distributive Law
Motivating the Three Laws
Multiplying by 1
Multiplying by 0
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The Equation $(-a)(-b) = ab$

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Summary   ***6. Exercises on Multiplication***
Worksheet ***6. Exercises on Multiplication***

Movie   ***6. Exercises on Multiplication***

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Exercise 6: $-a^3 b$
Exercise 7: $(-a)^2 b$
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Document: ***7. Fractions and the Operation of Division***

Summary   ***7. Fractions and the Operation of Division***

Movie   ***7. Fractions and the Operation of Division***

Introducing the Operation divides
Introducing The Notation b^{-1}
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The Cancellation Law for Multiplication
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Dividing by b^{-1} Is the Same as Multiplying by b
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When the Product of Numbers is Equal to Zero
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The equation $(bc)^{-1} = b^{-1}c^{-1}$
Dividing a Fraction by a Given Number
Cancelling in a Fraction
Inverting a Fraction
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Multiplying Two Fractions
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Document: ***8. Exercises on Multiplication and Division***

Summary   ***8. Exercises on Multiplication and Division***

Movie   ***8. Exercises on Multiplication and Division***

Worksheet ***8. Exercises on Multiplication and Division***

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Exercise 3: $\dfrac{\left(\dfrac{1}{2}\right)}{\left(\dfrac{2}{3}\right)}$

Exercise 4: $\dfrac{\left(\dfrac{\left(\dfrac{1}{2}\right)}{2}\right)}{3}$

Exercise 5: $\dfrac{\left(\dfrac{1}{\left(\dfrac{2}{3}\right)}\right)}{3}$

Exercise 6: $\dfrac{1}{a^{-1}}$

Exercise 7: $\dfrac{a}{b^{-1}}$

Exercise 8: $\dfrac{a^{-1}}{b^{-1}}$

Exercise 9: $\left(\dfrac{a^{-1}}{b^{-1}}\right)^{-1}$

Exercise 10: $\dfrac{2}{3} \times \dfrac{5}{7}$

Exercise 11: $\dfrac{2}{3} \div \dfrac{5}{7}$

Exercise 12: $\dfrac{2}{3} + \dfrac{5}{7}$

Exercise 13: $\dfrac{5}{6} - \dfrac{3}{10}$

Exercise 14: $\dfrac{a}{xy} + \dfrac{b}{xz}$

Exercise 15: $\dfrac{2}{\dfrac{1}{2} - \dfrac{1}{3}}$

Exercise 16: $\dfrac{\dfrac{1}{a} + \dfrac{1}{b}}{c}$

Exercise 17: $\dfrac{a}{\dfrac{1}{x} + \dfrac{1}{y}}$

Exercise 18: $(x^{-1} + y^{-1})^{-1}$

Exercise 19: $\left((x^{-1})^{-1} + (y^{-1})^{-1}\right)^{-1}$

Exercise 20: $\dfrac{2 + 13}{2 + 8}$

Exercise 21: $\left(\left(\dfrac{a}{b}\right)^{-1} + \left(\dfrac{c}{d}\right)^{-1}\right)^{-1}$

The Algebra Boot Camp Group

Document: 1. The Number Line

Summary   *1. The Number Line*

Movie   *1. The Number Line*

The Relation $<$
Two Important Facts About the Relation $<$
The Words "Positive" and "Negative"
Distance Between Numbers and Absolute Value

Document: 2. Expanding Algebraic Expressions

Summary   *2. Expanding Algebraic Expressions*
Worksheet *2. Expanding Algebraic Expressions*

Movie   *2. Expanding Algebraic Expressions*

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Exercise 10: $-4a(5x - 3y)$

![red icon] **Document:** *3. Expanding Algebraic Expressions that Contain Exponents*

Summary   *3. Expanding Algebraic Expressions that Contain Exponents*
Worksheet   *3. Expanding Algebraic Expressions that Contain Exponents*

Movie   *3. Expanding Algebraic Expressions that Contain Exponents*

Preliminary Discussion on Expansions
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Exercise 6: $(2x - y)(2x + y)$
Exercise 7: $(2x - 3y)^2$
Exercise 8: $(3x + 4yz)^2$
Exercise 9: $(3a^2 + 5b^3)^2$
Exercise 10: $(4x^4 - 3y^3)^2$
Exercise 11: $(3x^3 + 2y)^2$
Exercise 12: $(3x^3 - 2y)^2$
Exercise 13: $(3x^3 - 2y)(3x^3 + 2y)$
Exercise 14: $(4x^4 - 3y^3)(4x^4 + 3y^3)$
Exercise 15: $(3xy + 2u^2)^2$
Exercise 16: $(2x^3y - 5az^2)^2$
Exercise 17: $(4xy^4 - 3z)^2$
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Exercise 24: $(2x + 3y)^2 - (x - 5y)^2$
Exercise 25: $(x - y)(x + y)$
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Exercise 27: $(x - y)(x^2 + xy + y^2)$
Exercise 28: $(x - y)(x^3 + x^2y + xy^2 + y^3)$
Exercise 29: $(x + y)(x^2 + y^2)$
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Exercise 31: $(a + b + c)(a^2 + b^2 + c^2 - ab - ac - bc)$

![red icon] **Document:** *4. Introduction to Factorization*

Summary   *4. Introduction to Factorization*
Worksheet   *4. Introduction to Factorization*

Movie   *4. Introduction to Factorization*

Factors of an Integer
Terms and Factors of Polynomial Expressions
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Exercise 5: $a^2b^6x + a^5b^3y + a^3b^4xy$

Exercise 6: $x(x + y) + y(x + y)$
Exercise 7: $(a + 2b)x^2y + (a + 3b)xy^2$
Exercise 8: $(a + 2b)(a - b) + (a + 2b)(a - 5b)$
Exercise 9: $x^4y^3(x + y) - x^3y^5(x - y)$
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Document: *5. Factorization by Common Factors and Grouping*

Summary *5. Factorization by Common Factors and Grouping*
Worksheet *5. Factorization by Common Factors and Grouping*

Movie *5. Factorization by Common Factors and Grouping*

An Example to Illustrate the Method of Grouping
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Exercise 5: $(x + 2y)(x - 3y) - a(x - 2y)(x - 3y)$
Exercise 6: $a(x - 3y) - b(3y - x)$
Exercise 7: $ab^3(x + 2y)(x - 3y) - a^2b(x - 2y)(3y - x)$
Exercise 8: $(a + 3b)x - (3b + a)y$
Exercise 9: $(a + 3b)(a - 3b) + (3b + a)(3b - a)$
Exercise 10: $(a + 3b)xy + (3a + b)vy$

Document: *6. Factorizing a Difference of Squares*

Summary *6. Factorizing a Difference of Squares*
Worksheet *6. Factorizing a Difference of Squares*

Movie *6. Factorizing a Difference of Squares*

Exercise 1: $x^2 - 9$
Exercise 2: $4x^2 - 9y^2$
Exercise 3: $(a + 5b)^2 - 9y^2$
Exercise 4: $4(x - y)^2 - p^2$
Exercise 5: $16x^{16} - y^2$
Exercise 6: $2335^2 - 2334^2$
Exercise 7: $25x^2 - 9(x + 2y)^2$
Exercise 8: $4x^2 - 9(3x - y)^2$
Exercise 9: $25x^2 - (x + 2y)^2$
Exercise 10: $18x^2 - 2(2a - 5b)^2$
Exercise 11: $16x^{16} - 81y^4$
Exercise 12: $(x - 2y)^2 - 4(x - a)^2$
Exercise 13: $(x^2 + y^2)^2 - 4x^2y^2$
Exercise 14: $(2x + 3y)^4 - (x - y)^4$

Document: *7. Factorizing a Sum or Difference of Cubes*

Summary *7. Factorizing a Sum or Difference of Cubes*
Worksheet *7. Factorizing a Sum or Difference of Cubes*

Movie *7. Factorizing a Sum or Difference of Cubes*

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Exercise 4: $k^3 - 27$
Exercise 5: $125a^3 - b^3$
Exercise 6: $125a^3 - 8b^3$

Exercise 7: $250a^4 - 54ab^3$
Exercise 8: $a^6 - b^3$
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Sum of Cubes
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Exercise 10: $8 + m^3$
Exercise 11: $8 - m^3$
Exercise 12: $\dfrac{1}{8}x^3 - 125y^3$
Exercise 13: $27x^3 + \dfrac{8}{125}y^3$
Exercise 14: $27x^6 + \dfrac{8}{125}y^9$
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Exercise 16: $216x^{12}y^{18} + \dfrac{125}{216}z^{21}$
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Exercise 2: $8x^3 + 125y^3$
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Exercise 4: $a^4 + 16b^2$
Exercise 5: $\dfrac{1}{8}a^6 - 8b^3$
Exercise 6: $2ax^6 - 16a^4y^3$
Exercise 7: $a^6 - b^6$
Exercise 8: $a^6 + b^6$
Exercise 9: $64a^{12} + b^6$

Document: 8. Factorizing Quadratics

Summary **8. Factorizing Quadratics**
Worksheet **8. Factorizing Quadratics**

Movie **8. Factorizing Quadratics**

Introduction
More General Quadratics
Exercises on Guessing the Factors
Exercise 1: $64a^{12} + b^6$
Exercise 2: $6x^2 - 5x - 6$
Exercise 3: $6x^6 - 5x^3y^5 - 6y^{10}$
Exercise 4: $6u^2 - 7abu + 2a^2b^2$
Exercise 5: $-6x^2 - 35x + 6$
Exercise 6: $12x^2 + 6x - 90$
A Systematic Approach to Factorization of Quadratics (**See the movie for exercises on this topic**.)
Optional Appendix on Use of the Computer Algebra System in Scientific Notebook or Scientific Workplace

Document: 9. Assorted Exercises on Factorization

Summary **9. Assorted Exercises on Factorization**
Worksheet **9. Assorted Exercises on Factorization**

Movie **9. Assorted Exercises on Factorization**

The starting number of this set of exercises is 13 because these exercises are to be seen as a continuation of the exercises in movie titled **Factorizing Quadratics**.

Exercise 13: $x^4 - 16$
Exercise 14: $9a^2 - 25x^2y^4$
Exercise 15: $16x^{16} - y^2$
Exercise 16: $32x^{16} - 2y^2$
Exercise 17: $\dfrac{x^2}{9} - \dfrac{4y^2}{25}$
Exercise 18: $4x^2 - 9(2x - y)^2$
Exercise 19 $4x^2 - (2x - y)^2$
Exercise 20: $(5x - 7)^2 - (x - 5)^2$

Exercise 21: $9(x-y)^2 - 4(x+y)^2$

Exercise 22: $4(a+3b)^2 - 9(3b+2c)^2$

Exercise 23: $(3a+b)^2 - 5b(3a+b) - 6b^2$

Exercise 24: $15x^2 + 2x(x+4y) - (2x+5y)^2$

Exercise 25: $(2x^2+7x)^2 - 3(2x^2+7x) - 54$

Exercise 26: $(2x^2+x)^2 - 4(2x^2+x) + 3$

Exercise 27: $(a^2+3ab)^2 - 2b^2(a^2+3ab) - 8b^4$

Exercise 28: $24 - 10(3x^2-5x) - (3x^2-5x)^2$

Exercise 29: $3a^4 + 2a^2(2ab+b^2) - (2ab+b^2)^2$

Exercise 30: $12x^4 - 8x^2(y^2-xy) + (y^2-xy)^2$

Exercise 31: $(a+b)^2 - 4(a^2-b^2) - 12(a-b)^2$

Exercise 32: $a^3 + 8$

Exercise 33: $8x^3 + 125y^3$

Exercise 34: $64 - 27a^3b^6$

Exercise 35: $27a^3 - 8b^3$

Exercise 36: $27a^3 + 8b^3$

Exercise 37: $a^3 - \dfrac{1}{8}$

Exercise 38: $x^3 - \dfrac{64}{x^3}$

Exercise 39: $24x^4 + 81xy^3$

Exercise 40: $54c^4 - 128c$

Exercise 41: $(a+b)^3 + c^3$

Exercise 42: $a^3 - (b-c)^3$

Exercise 43: $(x-2y)^3 - y^3$

Exercise 44: $(1+y)^3 - (1-y)^3$

Exercise 45: $(2a+b)^3 - (a-2b)^3$

Exercise 46: $(2x-y)^3 - (x+3y)^3$

Exercise 47: $(x^2+x)^3 - 8$

Exercise 48: $(2a-b)x + (b-2a)y + 2a - b$

Document: **10. Sixty More Exercises on Factorization**

Summary   **10. Sixty More Exercises on Factorization**

Worksheet   **10. Sixty More Exercises on Factorization**

Movie   **10. Sixty More Exercises on Factorization**

Exercise 1: $mx + 9py - 3my - 3px$

Exercise 2: $2a^2m^2 - 2n^2b^2 + 4a^2n^2 - m^2b^2$

Exercise 3: $4a^2 + 25b^2 - 36c^2 - 20ab$

Exercise 4: $b^2 - 25 + 4ab + 4a^2$

Exercise 5: $24ab - 9a^2 + 1 - 16b^2$

Exercise 6: $4x^2 + 4yz - z^2 - 4y^2$

Exercise 7: $x^4 + 12x - 4x^2 - 9$

Exercise 8: $a^2 - b^2 + c^2 - d^2 - 2(ac+bd)$

Exercise 9: $4a^2 - 9c^2 + 12ab - 16 + 9b^2 - 24c$

Exercise 10: $4a^2 - 24cd - 4ab - 9c^2 - 16d^2 + b^2$

Exercise 11: $6cd - 20ab - d^2 + 4a^2 - 9c^2 + 25b^2$

Exercise 12: $4x(x+1) - (2y+1)(2y-1)$

Exercise 13: $b(b-2) - (a+1)(a-1)$

Exercise 14: $(3a+2b)(3a-2b) - 4c(3a-c)$

Exercise 15: $(x+1)(x-1) - y(y+2)$

Exercise 16: $b^2(a^2-4) - 4(ab-1)$

Exercise 17: $(x^2+1)(x^2-1) - x^2(2x-1)$

Exercise 18: $(ax+by)(ax-by) + 2by - 1$

Exercise 19: $2x^3 + x^2 - 8a^2x - 4a^2$

Exercise 20: $8 - 4a - 2a^2 + a^3$

Exercise 21: $8 - 4a - 2a^2 + a^3$

Exercise 22: $a^5 - a^3b^2 + a^2b^3 - b^5$

Exercise 23: $4 - 9x^2 - 4x^3 + 9x^5$

Exercise 24: $4x^5 - 9x^3 - 32x^2 + 72$

Exercise 25: $x^4 + x^2y^2 + y^4$

Exercise 26: $x^4 - 7x^2y^2 + y^4$

Exercise 27: $4x^4 + y^4$

Exercise 28: $(x^2 - 5x + 6)^2 - 2x(x^2 - 5x + 6)$

Exercise 29: $(b - c)(a^2 + 1) - (b - c)2a$

Exercise 30: $(x - 2)(2x + 5)^2 - (x - 2)^2$

Exercise 31: $a^3 - b^3 - (x^2 - ab)(a - b)$

Exercise 32: $a(a^2 - b^2) - 2b^2(a - b)$

Exercise 33: $a^3 - 1 + 3(a^2 - 1)$

Exercise 34: $a^2 - ab - 3a + 2b + 2$

Exercise 35: $x^2 + 3xy + 7x + 6y + 10$

Exercise 36: $6x^2 - 2xy - 5x + 3y - 6$

Exercise 37: $a^2 + 2ab + b^2 + a + b$

Exercise 38: $a^2 - 4ab + 4b^2 - 3a + 6b$

Exercise 39: $6a^2 - bc - 2b^2 - 2ac - ab$

Exercise 40: $2x^2 - 3xy - 2xz - 9y^2 + 6yz$

Exercise 41: $3x^2 - 5xy + 2y^2 + x^2y - xy^2$

Exercise 42: $1 + x - 2x^2 - x^3 + x^4$

Exercise 43: $x^4 - 2x^3 + 3x^2 - 4x + 2$

Exercise 44: $a^2 + a - b^2 + b$

Exercise 45: $4x^2 - 9y^2 - 2x + 3y$

Exercise 46: $4a^2 - 6ac - 9bc - 9b^2$

Exercise 47: $a^3 + a + b + b^3$

Exercise 48: $8x^3 - y^3 - 2x + y$

Exercise 49: $2x^3 - 3x^2 - 3x + 2$

Exercise 50: $3x^3 + 7x^2 - 14x - 24$

Exercise 51: $x^3 + x - \dfrac{1}{x} - \dfrac{1}{x^3}$

Exercise 52: $u(u - 2) - v(v + 2)$

Exercise 53: $4a^2(3b + 1) - 9b^2(2a + 1)$

Exercise 54: $(2x - y)y^2 - (2y - x)x^2$

Exercise 55: $x^2 + y^2 - 2(xy + yz - xz)$

Exercise 56: $ac(b^2 + 1) - b(a^2 + c^2)$

Exercise 57: $(a - 1)(a - 2) - (a - 3)(a + 20)$

Exercise 58: $a(b + c)(b - c) + bc(a + 1)(a - 1)$

Exercise 59: $a(bx^2 - axy - ay) + b(bxy - ay^2 + ax)$

Exercise 60: $1 - 3(1 - 2x) + 3(1 - 2x)(1 - 3x)$

Document: **11. An Optional Special Topic on Factorization**

Summary   **11. An Optional Special Topic on Factorization**

Worksheet   **11. An Optional Special Topic on Factorization**

Movie   **11. An Optional Special Topic on Factorization**

Exercise 1: $1 + a^3 + b^3 - 3ab$

Exercise 2: $1 + 8a^3 + b^3 - 6ab$

Exercise 3: $a^3 - 27 - 8b^3 - 18ab$

Exercise 4: $64 + \dfrac{a^3}{8} + 27b^3 - 18ab$

Exercise 5: $x^3 + 2 + 4 - 3\sqrt[3]{2}\,\sqrt[3]{4}\,x$

Exercise 6: $x^3 + 3 + 9 - 9x$

Exercise 7: $x^3 + 5 + 25 - 15x$

Exercise 8: $x^3 + 42 - 18x$

Exercise 9: $8x^9 + 36x^4 - 27x^3 + 8$

Exercise 10: $(a - b)^3 + (a + b)^3 + 8 - 6a^2 + 6b^2$

Exercise 11: $(u + 2v)^3 + (3u - v)^3 + 8 - 18u^2 - 30uv + 12v^2$

Exercise 12: $a^3 + b^3 + c^3 + d^3 - 3abc - 3abd - 3acd - 3bcd$

Exercise 13: $8 + 27u^3 + v^3 - w^3 - 18uv + 18uw + 6vw + 9uvw$

Exercise 14: $a^3 + b^3 + c^3 + 1 - 3abc - 3ab - 3ac - 3bc$

Exercise 15: $a^3 + b^3 + 8 + 1 - 6ab - 3ab - 6a - 6b$

Exercise 16: $a^3 + b^3 + 2 + 4 - 3\sqrt[3]{2}\, ab - 3\sqrt[3]{4}\, ab - 6a - 6b$

Exercise 17: $a^3 + b^3 + 2 + 4 - 3\sqrt[3]{2}\, ab - 3\sqrt[3]{4}\, ab - 6a - 6b$

12. Algebraic Fractions

Summary   **12. Algebraic Fractions**
Worksheet **12. Algebraic Fractions**

Movie   **12. Algebraic Fractions**

Exercise 1: $\dfrac{371}{707}$

Exercise 2: $\dfrac{2 + \frac{2}{3}}{\frac{5}{6}}$

Exercise 3: $\dfrac{\frac{2}{3}}{\frac{1}{6} + \frac{5}{3}}$

Exercise 4: $\dfrac{1}{\frac{1}{x} + \frac{1}{y}}$

Exercise 5: $\dfrac{x^2 - 5x}{x^2 - 3x - 10}$

Exercise 6: $\dfrac{x^3 - 8}{x^2 + 2x + 4}$

Exercise 7: $\dfrac{x^4 - 27x}{x^2 + x - 12}$

Exercise 8: $\dfrac{\frac{1}{x+y} - \frac{1}{x-y}}{\frac{y}{x^2 - y^2}}$

Exercise 9: $\dfrac{x^3 - 8}{5x^2 - 9x - 2}$

Exercise 10: $\dfrac{6x^2 - 6y^2 - 5xy}{8x^3 - 27y^3}$

Exercise 11: $\dfrac{\frac{a-3}{a-4} - \frac{a-4}{a-3}}{\frac{1}{a-3} - \frac{1}{a-4}}$

Exercise 12: $\dfrac{x^3 - 3x^2 y - y^3 + 3y^2 x}{x - y}$

Exercise 13: $\dfrac{a^4 - b^4}{a^3 - b^3 + ab^2 - ba^2}$

Exercise 14: $\dfrac{x^3 - 125}{x^2 + x - 6} \times \dfrac{3x^2 - 4x - 4}{3x^2 - 13x - 10} \times \dfrac{x + 3}{x^2 + 5x + 25}$

Exercise 15: $\left(\dfrac{\frac{3x^2 - 5x - 12}{4x^2 + 13x + 10}}{\frac{12x^2 + x - 20}{3x^2 + 9x + 27}} \right) \left(\dfrac{32x^3 - 50x}{x^3 - 27} \right)$

Exercise 16: $\dfrac{\left(\frac{(a-b)^2 - c^2}{ab - a^2 - ac} \right) \left(\frac{a^2 - (b-c)^2}{(a-b+c)^2} \right)}{\frac{(b+c)^2 - a^2}{(a+c)^2 - b^2}}$

Exercise 17: $\dfrac{3}{x+1} + \dfrac{1}{x-3}$

Exercise 18: $\dfrac{1}{x-2} - \dfrac{1}{x+4} + \dfrac{6}{(x+4)^2}$

Exercise 19: $\dfrac{5}{3(x-1)^2} + \dfrac{1}{9(x-1)} - \dfrac{1}{9(x+2)}$

Exercise 20: $\dfrac{x+2}{(2x-1)(x+3)} + \dfrac{3x+1}{(2x-1)(x-2)} - \dfrac{6x+2}{(2x-1)(x+3)(x-2)}$

Exercise 21: $\dfrac{3}{x-3} - \dfrac{2}{x+3} + \dfrac{9}{9 - x^2}$

Exercise 22: $\dfrac{1 + 2x}{1 + 2x + 4x^2} - \dfrac{2x}{2x - 1} - \dfrac{8x^3}{1 - 8x^3}$

Exercise 23: $\dfrac{x+1}{x+2} + 2\left(\dfrac{x-1}{x+2} \right)^2 + \dfrac{x^2 - 1}{2x^2 + 3x - 2} - \dfrac{7x^3 - 7x^2 + 2x - 2}{(2x^2 + 3x - 2)(x+2)}$

Exercise 24: $\left(\dfrac{a}{a+x} - \dfrac{a+x}{a+3x}\right)\left(\dfrac{x}{x+a} - \dfrac{x+a}{x-a}\right)$

Exercise 25: $\dfrac{1 - \dfrac{2ab}{a^2+b^2}}{\dfrac{a^3-b^3}{a-b} - 3ab}$

Exercise 26: $\left(\dfrac{a}{b} - \dfrac{a+b}{2a}\right)\left(\dfrac{a^2+ab+b^2}{a^2+ab} - \dfrac{a^2-ab+b^2}{ab-a^2}\right)$

Exercise 27: $\left(\dfrac{x+y}{2(x-y)} - \dfrac{x-y}{2(x+y)} + \dfrac{2y^2}{y^2-x^2}\right)\left(1 + \dfrac{y}{x}\right)$

Exercise 28: $\left(\dfrac{a}{a+b} - \dfrac{b}{b-a} + \dfrac{2ab}{a^2-b^2}\right)\left(1 - \left(\dfrac{2b}{a+b}\right)^2\right)$

Exercise 29: $\dfrac{x^3 - \dfrac{1}{x^3} - 3\left(x - \dfrac{1}{x}\right)}{x - \dfrac{1}{x}}$

Exercise 30: $\dfrac{a}{a^2-(b-c)^2} - \dfrac{b}{b^2-(a-c)^2}$

Exercise 31: $\dfrac{bc-a^2}{(a-b)(c-a)} + \dfrac{ca-b^2}{(b-c)(a-b)} + \dfrac{ab-c^2}{(c-a)(b-c)}$

Exercise 32: $\dfrac{2a-1}{(a-1)(2a-3)} - \dfrac{8(a-1)}{(1-2a)(3-2a)} - \dfrac{2a-3}{(2a-1)(1-a)}$

Exercise 33: $\dfrac{a}{1-b} - \dfrac{a}{1+b}$

▬ 13. Division of Polynomials

Summary   13. Division of Polynomials

Movie   🎬 13. Division of Polynomials

▬ 14. Solving Linear Equations

Summary   14. Solving Linear Equations
Worksheet   14. Solving Linear Equations

Movie   🎬 14. Solving Linear Equations

Introducing Linear Equations
Some Exercises on Linear Equations
Exercise 1: $3x - 2 = 5(x+4)$
Exercise 2: $(x-3)(x+1) = (x-5)(x+2)$
Exercise 3: $\dfrac{2x+3}{4} - \dfrac{x-1}{3} = 1$
Exercise 4: $\dfrac{2}{11-x} = \dfrac{1}{x+2}$
Exercise 5: $a(x-1) + b(2x+1) + c = 0$
Exercise 6: $c = \dfrac{a+b}{1+ab}$
Exercise 7
Exercise 8
The Relationship Between Speed, Distance, and Time
Exercise 9
Exercise 10
Exercise 11
Exercise 12

▬ 15. Using Factors to Solve Equations

Summary   15. Using Factors to Solve Equations
Worksheet   15. Using Factors to Solve Equations

Movie   🎬 15. Using Factors to Solve Equations Free Demo Movie

Exercise 1: $(2x-6)^4(x-2)^{77} = 0$
Exercise 2: $2x^2 + 5x - 12 = 0$
Exercise 3: $7x - 3 - 2x^2 = 0$
Exercise 4: $(2x-7)^2 = 0$
Exercise 5: $x(x+3) = 10$
Exercise 6: $(2x-1)(x-1) = 6$
Exercise 7: $(x^2-x-6)^3(3x^2-13x-10)^{100} = 0$

Exercise 8: $2x^3 - 3x^2 - 2x + 3 = 0$
Exercise 9: $x^4 - 13x^2 + 36 = 0$
Exercise 10; $x^4 - 5x^2 - 36 = 0$
Exercise 11: $(x+4)(x-1) = 6$

16. Solving Square-Complete Quadratic Equations

Summary **16. Solving Square-Complete Quadratic Equations**
Worksheet **16. Solving Square-Complete Quadratic Equations**

Movie **16. Solving Square-Complete Quadratic Equations**

Linear, Quadatic, and Cubic Equations
Quadratic Equations that Are Square Complete

Examples of Square-Complete Quadratic Equations
Example 1: $x^2 - 9 = 0$
Example 2: $x^2 - 7 = 0$
Example 3: $3x^2 - 5 = 0$
Example 4: $x^2 + 9 = 0$
Example 5: $(x-3)^2 = 16$
Example 6: $(2x-5)^2 - 8 = 0$

Exercises on the Solution of Quadratic Equations That Are in Square Complete Form
Exercise 1: $(x-1)^2 = 1$
Exercise 2: $\left(x + \frac{3}{5}\right)^2 = \frac{4}{25}$
Exercise 3: $(x+2)^2 = 6$
Exercise 4: $\left(x - \frac{2}{3}\right)^2 = \frac{7}{9}$
Exercise 5: $\left(x - \frac{4}{7}\right)^2 = 0$
Exercise 6: $\left(x^2 - \frac{13}{2}\right)^2 = \frac{25}{4}$
Exercise 7: $(x+1)^2 = -9$
Exercise 8: $(x^2 - 2)^2 = 49$
Exercise 9: $(x^2 - 2)^2 = 48$
Exercise 10: $(x^2 - 3x - 4)^2 = 36$
Exercise 11: $(x-2)^2 = 9(2x+3)^2$
Exercise 12: $(x-2)^2 = -4(x-3)^2$
Exercise 13: $\left(\frac{2}{x-3} - 5\right)^2 = 9$
Exercise 14: $\left(x + \frac{1}{x}\right)^2 = 4$
Exercise 15: $\left(\frac{x-2}{x+3}\right)^2 = 16$

17. Solving Quadratic Equations

Summary **17. Solving Quadratic Equations**
Worksheet **17. Solving Quadratic Equations**

Movie **17. Solving Quadratic Equations**

Strategy for Solving Quadratic Equations
Review of the Solution of a Square Complete Equation

Exercises on Completing the Square: Getting Rid of the Middle Term
Exercise 1: $x^2 - 4x - 2 = 0$
Exercise 2: $x^2 + 6x + 6 = 0$
Exercise 3: $x^2 - 3x - 1 = 0$
Exercise 4: $x^2 + 3x + 1 = 0$
Exercise 5: $3x^2 + 6x - 2 = 0$
Exercise 6: $2x^2 + 3x - 1 = 0$
Exercise 7: $2x^2 + 3x + 2 = 0$
Exercise 8: $3x^2 - 5x - 1 = 0$
Exercise 9: $x^2 - 6x + 8 = 0$
Exercise 10: $x^2 + 6x + 9 = 0$
Exercise 11: $x^2 - 6x + 10 = 0$
Exercise 12: $x^4 + 10x^2 + 14 = 0$
Exercise 13: $x^3 - 3x^2 - 3x = 0$
Exercise 14: $(x^2 + x)^2 - 8(x^2 + x) + 12 = 0$
Exercise 15: $(2x^2 - 5x)^2 - (2x^2 - 5x) - 6 = 0$

Exercise 16: $x + \dfrac{2}{x} = 4$

Exercise 17: $2(x^2 - x) = \dfrac{1}{x^2 - x} + \dfrac{7}{2}$

Exercise 18: $\dfrac{x^2 - 5x}{6} + \dfrac{5}{3} + \dfrac{4}{x^2 - 5x} = 0$

18. The Formula for Solving Quadratic Equations

Summary   *18. The Formula for Solving Quadratic Equations*
Worksheet *18. The Formula for Solving Quadratic Equations*

Movie   *18. The Formula for Solving Quadratic Equations*

Statement of the Quadratic Formula
Deriving the Quadratic Formula
The Discriminant of a Quadratic
Summary of the Preceding Remarks
Some Examples to Illustrate Use of the Quadratic Formula
Example 1: $3x^2 - 5x + 1 = 0$
Example 2: $12x^2 - 17x + 6 = 0$
Example 3: $12x^2 - 17x + 6 = 0$

19. Exercises on the Quadratic Formula

Summary   *19. Exercises on the Quadratic Formula*
Worksheet *19. Exercises on the Quadratic Formula*

Movie   *19. Exercises on the Quadratic Formula*

A Non-Exercise on the Quadratic Formula
Some Exercises on the Quadratic Formula
Exercise 1: $x^2 - ax + 9 = 0$
Exercise 2: $x^2 + 12x + a = 0$
Exercise 3: $x^2 - (m + 1)x + m^2 - 1 = 0$
Exercise 4: $(a + b)x^2 - (a + 2b)x - 2a = 0$
Exercise 5: $(a + b + c)x^2 - 2(a + b)x + a + b - c = 0$
Exercise 6: $(a^2 - 4bc)x^2 + 4(b + c)x - 4 = 0$
Exercise 7: $x^2 - ax - bx - ab + a^2 + b^2 = 0$
Exercise 8: $x^2 - ax - bx - cx - ab - bc - ac + a^2 + b^2 + c^2 = 0$
Exercise 9: $at^2 + bt - ax^2 - bx - mt + mx = 0$
Exercise 10: $ax^2 + bx + c - mx + mp - q = 0$

20. Solving Fractional Equations

Summary   *20. Solving Fractional Equations*
Worksheet *20. Solving Fractional Equations*

Movie   *20. Solving Fractional Equations*

Exercise 1: $\dfrac{x + 3}{x - 3} + \dfrac{x - 3}{x + 3} = \dfrac{5}{2}$

Exercise 2: $\dfrac{3}{1 - u} + \dfrac{6}{1 + u} = \dfrac{7}{u - 2}$

Exercise 3: $\dfrac{4 - u}{4 - u^2} = \dfrac{u}{u - 2} + 2$

Exercise 4: $\dfrac{u}{1 - u} - \dfrac{3}{u^2 - 1} = \dfrac{u + 4}{2(u + 1)}$

Exercise 5: $\dfrac{2}{5(x - 2)} + \dfrac{x + 1}{4 - x^2} = \dfrac{1 - x}{x + 2}$

Exercise 6: $1 = \dfrac{9}{x + 5} + \dfrac{3}{x - 2} + \dfrac{1 - 3x}{(x - 2)(x + 5)}$

Exercise 7: $\dfrac{3}{x + 3} - 1 = \dfrac{2x}{x - 2} - \dfrac{5(x + 2)}{(x + 3)(x - 2)}$

21. Solving Equations that Contain Radicals

Summary   *21. Solving Equations that Contain Radicals*
Worksheet *21. Solving Equations that Contain Radicals*

Movie   *21. Solving Equations that Contain Radicals*

Exercise 1: $\sqrt[3]{x} = 2$

Exercise 2: $\sqrt[5]{x} - 2 = 0$

Exercise 3: $\sqrt{x + 6} = x$

Exercise 4: $\sqrt[3]{x^2 - 6x + 20} = 3$

Exercise 5: $\sqrt{\sqrt[3]{x} + 4} = 3$

Exercise 6: $\sqrt[3]{x^3 + 4x^2 + 2x + 1} = x + 1$

Exercise 7: $x - 3\sqrt{x} + 2 = 0$

Exercise 8: $\sqrt{x - 4} + x = 16$

Exercise 9: $\sqrt{x - 2} + \sqrt{3x - 1} = \sqrt{5}$

Exercise 10: $\sqrt{x + 4} = \sqrt{x - 7} - 1$

Exercise 11: $\sqrt{x - 7} = \sqrt{x + 4} - 1$

Exercise 12: $x - \sqrt{x + 7} = -1$

Exercise 13: $\sqrt[5]{x^2 - x - 5} = 1$

Exercise 14: $\sqrt[3]{x^3 + x^2 - 3x + 9} = x + 1$

Exercise 15: $\sqrt{x + 6} + \sqrt{2x + 5} = 3$

Exercise 16: $\sqrt{x + 1} + \sqrt{x + 6} = \sqrt{x + 22}$

Exercise 17: $\dfrac{x}{\sqrt{x - 2}} + \sqrt{x - 2} = 5$

Exercise 18: $\dfrac{1}{2x + \sqrt{x^2 - 1}} + \dfrac{1}{2x - \sqrt{x^2 - 1}} = 1$

Exercise 19: $\sqrt[3]{(x - 3)^2} + 5 = 6\sqrt[3]{x - 3}$

22. Systems of Equations

Summary   22. Systems of Equations

Worksheet 22. Systems of Equations

Movie   22. Systems of Equations

Linear Systems of Equations

Exercise 1: $x - y = 2$ and $2x + y = 7$

Exercise 2: $2x + y = 7$ and $3x - 4y = -5$

Exercise 3: $3x + 2y - 4z = -12$ and $2x - 3y - 2z = -1$ and $5x + 4y - 2z = -2$

Exercise 4: $x - 2y + 3z = 2$ and $2x - 3y + 4z = 5$ and $x - y + z = 3$

Exercise 5: $x - 2y + 3z = 2$ and $2x - 3y + 4z = 5$ and $x - y + z = 4$

Linear-Quadratic Systems of Equations

Exercise 1: $x - y + z = 4$ and $x^2 + y^2 = 25$

Exercise 2: $x - y = 2$ and $x^2 - xy - y^2 = 4$

Exercise 3: $2x - y = 3$ and $x^2 - xy + 2y^2 = 4$

Exercise 4: $2x - 3y = 2$ and $2x^2 - 5xy + 4y^2 = 8$

Exercise 5: $2x + 3y = 5$ and $2x^2 + xy - x - y = 1$

Exercise 6: $3x - 2y - 2 = 0$ and $x^2 - 4y + 3x = 5$

Exercise 7: $x^2 + y^2 - x - y = 6$ and $x^2 + y^2 + x + 3y = 0$

Exercise 8: $x^2 + y^2 - 2x + y = 1$ and $5x^2 + 5y^2 - 6x + 7y = 5$

Exercise 9: $x^2 + y^2 + 2x + 7y + 16 = 0$ and $x^2 + y^2 - 4x - 2y + 1 = 0$

Additional Exercises on Linear-Quadratic Systems

Exercise 1: $y = mx + q$ and $y = ax^2 + bx + c$

Exercise 2: $y = at^2 + bt + c + m(x - t)$ and $y = ax^2 + bx + c$

Exercise 3: $x^2 + y^2 = r^2$ and $y = mx + b$

Exercise 4: $x^2 + y^2 = r^2$ and $y = mx + b$

23. Introduction to Inequalities

Summary   23. Introduction to Inequalities

Movie   23. Introduction to Inequalities

Exercise 4

Adding Smaller Numbers gives Less than Adding Larger Numbers

The Equivalence of the Statements $a < b$ and $-a > -b$

Inequalities and Subtraction

Multiplying Positive Numbers and Multiplying Negative Numbers

Multiplying Both Sides of an Inequality by a Positive Number

Comparing Positive Numbers and their Squares

Multiplying Both Sides of an Inequality by a Negative Number

Distance Between Numbers and Absolute Value
Absolute Value and Square Roots
Absolute Value of a Product
Absolute Value of a Sum

24. Exercises on Linear Inequalities

Summary   24. Exercises on Linear Inequalities
Worksheet 24. Exercises on Linear Inequalities

Movie   24. Exercises on Linear Inequalities

1. Simple Linear Inequalities

Exercise 1a: $3x + 2 \leq 17$
Exercise 1b: $-3x + 2 \leq 17$
Exercise 1c: $3|x| + 2 \leq 17$
Exercise 1d: $|x - 3| \leq 7$
Exercise 1e: $|x - 1| \leq -2$
Exercise 1f: $|x + 2| \geq -2$
Exercise 1g: $|x + 2| \geq 2$
Exercise 1h: $3 \leq |x + 1| \leq 7$

2. More Difficult Linear Inequalities

Exercise 2a: $|x - 1| < |x - 5|$
Exercise 2b: $|x - 1| < 2|x - 5|$
Exercise 2c: $|2x - 3| < |6 - x|$
Exercise 2d: $|x - 1| < 2|x - 5| + |x - 2|$
Exercise 2e: $|x - 1| < 2|x - 5| - |x - 2|$
Exercise 2f: $||x| - 5| < |x - 6|$
Exercise 2g: $||x| - 5| < 2|x - 6|$
Exercise 2h: $||x| - 5| < ||x| - 6|$
Exercise 3: $|x - y| \leq |x| + |y|$
Exercise 4: $||x| - |y|| \leq |x - y|$

25. Exercises on Nonlinear Inequalities

Summary   25. Exercises on Nonlinear Inequalities
Worksheet 25. Exercises on Nonlinear Inequalities

Movie   25. Exercises on Nonlinear Inequalities

Exercise 1: $(x - 1)(2 - x)(x + 3) \leq 0$

Exercise 2: $\dfrac{x - 3}{x + 2} \geq 0$

Exercise 3: $\dfrac{(x + 2)^2}{x - 3} \geq 0$

Exercise 4: $\dfrac{(x - 1)(x - 3)^2}{(x - 2)(x - 5)} \geq 0$

Exercise 5: $\dfrac{(x^2 - 4x + 5)(x - 2)}{x - 1} \leq 0$

Exercise 6: $\dfrac{\sqrt[3]{x - 4}}{(x - 1)(x - 2)} \geq 0$

Exercise 7: $\dfrac{x - 3}{x + 1} \geq 1$

Exercise 8: $\dfrac{3}{x + 2} \geq \dfrac{2}{3 - x}$

Exercise 9: $\dfrac{1}{12(x - 1)} + \dfrac{21}{4(x + 1)} \geq \dfrac{16}{3(x + 2)} + \dfrac{9}{2(x + 1)^2}$

Exercise 10: $ab \leq \dfrac{a^2}{2} + \dfrac{b^2}{2}$

Exercise 11: $\dfrac{c}{1 + c} < \dfrac{a}{1 + a} + \dfrac{b}{1 + b}$

26. Solving Cubic Equations

Summary   26. Solving Cubic Equations

Movie   26. Solving Cubic Equations

Preamble

Making of the Coefficient of x^3 to be Equal to 1
Getting Rid of the Term in x^2

Finding the Numbers *p* **and** *q*

The Case $4a^3 + 27a^2 > 0$

The Case $4a^3 + 27a^2 = 0$

The Case $4a^3 + 27a^2 < 0$

The Algebra of Exponents Group

1. Introduction to Positive Integer Exponents

Summary   *1. Introduction to Positive Integer Exponents*

Movie   *1. Introduction to Positive Integer Exponents*

Introducing the Notation of Exponents

The Laws of Positive Integer Exponents

The Law $(a^m)(a^n) = a^{m+n}$

The Law $\dfrac{a^m}{a^n} = a^{m-n}$

The Law $(a^m)^n = a^{mn}$

The Law $a^n b^n = (ab)^n$

The Law $\dfrac{a^n}{b^n} = \left(\dfrac{a}{b}\right)^n$

The Expression $a^m b^n$

2. Exercises on Positive Integer Exponents

Summary   *2. Exercises on Positive Integer Exponents*

Worksheet   *2. Exercises on Positive Integer Exponents*

Movie   *2. Exercises on Positive Integer Exponents*

Exercise 1: $x^3 x^4$

Exercise 2: $x^3 y^4$

Exercise 3: $\dfrac{a^{73}}{a^{68}}$

Exercise 4: $(3x^2)(2x^3)$

Exercise 5: $(xy^2)(x^3 y^4)$

Exercise 6: $(xy^2)^3$

Exercise 7: $(4x^4)^2$

Exercise 8: $\dfrac{(3a^4)^5}{(9a^2)^3}$

Exercise 9: $5^2 - 3^2$

Exercise 10: $5^2 - (-3)^2$

Exercise 11: $5^2 + 3^2$

Exercise 12: $(5 - 3)^2$

Exercise 13: $(5 + 3)^2$

Exercise 14: $(2(-3x^3))^2$

Exercise 15: $(2(-3x^3))^3$

Exercise 16: $(-2(-3x^3))^3$

Exercise 17: $((-2)(-3x^3))^3$

Exercise 18: $(-x^2 y^3)^3$

Exercise 19: $\left((-x)^2 y^3\right)^3$

Exercise 20: $(a^m)^n = (a^n)^m$

3. Introduction to More General Exponents

Summary   *3. Introduction to More General Exponents*

Movie   *3. Introduction to More General Exponents*

A Quick Review of Positive Integer Exponents

An Introduction to More General Exponents

The Expression a^0

The Relationship Between a^x and a^{-x}

The Expression a$^{1/x}$

The Expression 1^x

The Expression 0^x for $x > 0$

The Expression 0^0 is Undefined

4. Working with the Rules for Exponents

Summary **4. Working with the Rules for Exponents**

Movie **4. Working with the Rules for Exponents**

Example 1: $\dfrac{a^{-x}b}{c} = \dfrac{b}{a^{x}c}$

Example 2: $\dfrac{b}{a^{-x}c} = \dfrac{ba^{x}}{c}$

Example 3: $\dfrac{2^{-3}}{3^{-2}} = \dfrac{3^{2}}{2^{3}}$

Example 4: $\left(\dfrac{a^{-x}}{b^{-y}}\right)^{-p} = \dfrac{a^{px}}{b^{py}}$

Example 5: $a^{3/7} = \sqrt[7]{a^{3}}$

Example 6: $a^{-3/7} = \dfrac{1}{\left(\sqrt[7]{a}\right)^{3}}$

5. Exercises on General Integer Exponents

Summary **5. Exercises on General Integer Exponents**
Worksheet **5. Exercises on General Integer Exponents**

Movie **5. Exercises on General Integer Exponents**

Exercise 1: 2^{-5}

Exercise 2: $(3^{-2})(2^{3})$

Exercise 3: $a^{-4}b^{3}$

Exercise 4: $\dfrac{a^{-4}b^{-5}}{a^{-6}b^{-7}}$

Exercise 5: $\left(\dfrac{a}{b^{-2}}\right)^{-3}(a^{2}b^{-4})$

Exercise 6: $\left(\left(\dfrac{a^{-3}b^{2}}{a^{2}b^{-4}}\right)^{-4}\right)^{-2}$

Exercise 7: $\dfrac{(a^{3}b^{-2}c^{0})^{-4}}{(ab^{-3}c^{2})^{2}}$

Exercise 8: $\left(\dfrac{(ab^{2}c^{-4})^{3}(a^{-3}b^{2}c^{-4})^{-2}}{(a^{4}b^{5}c^{-6})^{2}}\right)^{-2}$

Exercise 9: $(a^{-1}+b^{-1})^{-1}$

Exercise 10: $\left(\dfrac{a^{n}+a^{-n}}{2}\right)^{2} - \left(\dfrac{a^{n}-a^{-n}}{2}\right)^{2}$

Exercise 11: $\left(\dfrac{a^{m}+a^{-m}}{2}\right)\left(\dfrac{a^{n}-a^{-n}}{2}\right) + \left(\dfrac{a^{m}-a^{-m}}{2}\right)\left(\dfrac{a^{n}+a^{-n}}{2}\right)$

Exercise 12: $\dfrac{x^{-1}y^{-1}}{y^{-1}-x^{-1}} + \dfrac{x^{-1}y^{-1}}{y^{-1}+x^{-1}}$

6. Exercises on Fractional Exponents

Summary **6. Exercises on Fractional Exponents**
Worksheet **6. Exercises on Fractional Exponents**

Movie **6. Exercises on Fractional Exponents**

Exercise 1: $32^{3/5}$

Exercise 2: $(-32)^{3/5}$

Exercise 3: $(-32)^{4/5}$

Exercise 4: $-32^{4/5}$

Exercise 5: $(-32)^{-4/5}$

Exercise 6: $\left(\dfrac{8a^{3}b^{6}}{27x^{27}}\right)^{4/3}$

Exercise 7: $\left(\dfrac{16a^{16}}{81b^{12}}\right)^{-3/4}$

Exercise 8: $\left(\dfrac{\sqrt[3]{a^{6}b^{3}c^{-9}}}{\sqrt[5]{a^{10}b^{-5}c^{-25}}}\right)^{2}$

Exercise 9: $\dfrac{a^{2/3} a^{5/12}}{\left(\sqrt[3]{a}\,\right)^{1/4}}$

Exercise 10: $\dfrac{\left(\sqrt{a^2 - b^2}\,\right)(a+b)^{1/2}}{(a-b)^{-3/2}}$

Exercise 11: $\left(\sqrt{96}\,\right)\left(\sqrt{6}\,\right)$

Exercise 12: $\sqrt{41^2 - 40^2}$

Exercise 13: $\sqrt[3]{172^2 - 171^2}$

Exercise 14: $\left(5\sqrt{3} + 2\sqrt{7}\,\right)\left(5\sqrt{3} - 2\sqrt{7}\,\right)$

Exercise 15: $\left(\sqrt[3]{3} - \sqrt[3]{2}\,\right)\left(\sqrt[3]{9} + \sqrt[3]{6} + \sqrt[3]{4}\,\right)$

Exercise 16: $(a^{1/3} - b^{1/3})(a^{2/3} + a^{1/3}b^{1/3} + b^{2/3})$

Exercise 17: $(a^{1/5} - b^{1/5})(a^{4/5} + a^{3/5}b^{1/5} + a^{2/5}b^{2/5} + a^{1/5}b^{3/5} + b^{4/5})$

Exercise 18

    Exercise 18a: $\dfrac{\sqrt{3} - \sqrt{2}}{\sqrt{5}}$

    Exercise 18b: $\dfrac{\sqrt{3} - \sqrt{2}}{\sqrt{3} + \sqrt{2}}$

Exercise 19

    Exercise 19a: $\dfrac{2\sqrt{3}}{3}$

    Exercise 19b: $\dfrac{\sqrt{t} - \sqrt{x}}{t - x}$

    Exercise 19c: $\dfrac{\sqrt[3]{t} - \sqrt[3]{x}}{t - x}$

    Exercise 19d: $\dfrac{\sqrt[3]{t} - \sqrt[3]{x}}{\sqrt{t} - \sqrt{x}}$

Exercise 20: $\dfrac{2}{3}x^{-1/3}(6-x)^{1/3} - x^{2/3}\left(\dfrac{1}{3}\right)(6-x)^{-2/3}$

Exercise 21: $\dfrac{-x^{1/3}(6-x)^{2/3} - (4-x)\left(\dfrac{1}{3}x^{-2/3}(6-x)^{2/3} - \dfrac{2}{3}x^{1/3}(6-x)^{-1/3}\right)}{x^{2/3}(6-x)^{4/3}}$

Exercise 22: $t = x$

    Exercise 22a: $\dfrac{t^{2/3} - x^{2/3}}{t^{1/3} - x^{1/3}}$

    Exercise 22b: $\dfrac{t^{5/3} - x^{5/3}}{t^{1/3} - x^{1/3}}$

    Exercise 22c: $\dfrac{t^{5/3} - x^{5/3}}{t^{2/3} - x^{2/3}}$

    Exercise 22d: $\dfrac{t^{5/7} - x^{5/7}}{t - x}$

Exercise 23: $\left(a^x\right)^{1/x} = a$

Exercise 24: $\dfrac{a^x}{b^x} = \left(\dfrac{a}{b}\right)^x$.

▬ *The Algebra of Logarithms Group*

▬ *1. Introduction to Logarithms*

Summary   **1. Introduction to Logarithms**

Movie   🎬 **1. Introduction to Logarithms**

Exponential Graphs
Introducing the Definition of a Logarithm
Some Examples of Logarithms
Example 1: $\log_2 8$
Example 2: $\log_5 1$
Example 3: $\log_a w$
Example 4: $\log_c y$
Example 5: $\log_2(-4)$
Example 6: $\log_3\left(\dfrac{1}{9}\right) = -2$
Example 7: Can we have the logarithm of a negative number?

Example 8: $\log_4 8$

Example 9: $\log_{\sqrt{32}}\left(\dfrac{1}{\sqrt[3]{4}}\right)$

2. Elementary Facts About Logarithms

Summary **2. Elementary Facts About Logarithms**

Movie **2. Elementary Facts About Logarithms**

Review of the Definition of a Logarithm
Some Elementary Exercises on Logarithms
Exercise 1: $\log_{16} 32$

Exercise 2: $\log_{\sqrt{32}}\left(\dfrac{1}{\sqrt[3]{4}}\right)$

Exercise 3: $\log_a a$

Exercise 4: $\log_a\left(\dfrac{1}{a}\right)$

Exercise 5: $\log_a(a^p)$

Exercise 6: $a^{(\log_a x)}$

Exercise 7: $\log_a 1$

3. The Laws of Logarithms

Summary **3. The Laws of Logarithms**

Movie **3. The Laws of Logarithms**

Preliminary Note
Logarithm of a Product
Logarithm of a Quotient
Logarithm of One Number to the Power of Another

4. Exercises on the Laws of Logarithms

Summary **4. Exercises on the Laws of Logarithms**
Worksheet **4. Exercises on the Laws of Logarithms**

Movie **4. Exercises on the Laws of Logarithms**

Exercise 1
    Exercise 1a: $\log_4 8$
    Exercise 1b: $\log_{\sqrt{8}} 16$

    Exercise 1c: $\log_{\sqrt[3]{32}}\left(\dfrac{1}{\sqrt[5]{16}}\right)$

Exercise 2: $\log_a x = 2$ and $\log_a y = 3$
    Exercise 2a: $\log_a(xy)$
    Exercise 2b: $\log_a\left(\dfrac{1}{x}\right)$
    Exercise 2c: $\log_a\left(\sqrt[3]{x}\,\sqrt[5]{y}\right)$
    Exercise 2d: $\log_a\left(x^{\log_a(y^3)}\right)$
Exercise 3: $\log_{10} 2 = u$ and $\log_{10} 3 = v$
    Exercise 3a: $\log_{10} 8$
    Exercise 3b: $\log_{10}(0.75)$
    Exercise 3c: $\log_{10}\sqrt{3}$
    Exercise 3d: $\log_{10} 5$
Exercise 4: $\log_{12} 3 = u$
Exercise 5: $\log_{60} 3 = u$
Exercise 6: $\log_3(-3)$
Exercise 7
    Exercise 7a: $\log_2(x-3) = 3$
    Exercise 7b: $\log_x(x+6) = 2$
    Exercise 7c: $\log_2(x-3) - \log_2(x-5) = 1$
    Exercise 7d: $\log_2(x-3) - \log_2(x-5) = -1$
    Exercise 7e: $\log_7(6x+1) + \log_7(x+2) = 1$
    Exercise 7f: $\log_{(x-1)}(25x^2 - 10x + 1) = 2$
    Exercise 7g: $\log_x(x+4) + 2\log_x(x-3) - \log_x(x-2) = 2$

Exercise 8:

Exercise 8a: $\log_{10}1000$

Exercise 8b: $\log_2 3$

Exercise 8c: $\log_{10}2$

Exercise 8d: $\log_{\sqrt[3]{4}}\left(\sqrt[7]{64}\right)$

Exercise 8e: $\log_{40}\left(\dfrac{1}{2\sqrt{10}}\right)$

Exercise 8f; $\log_{40}250$

5. Changing Base

Summary   *5. Changing Base*

Worksheet *5. Changing Base*

Movie   *5. Changing Base*

Changing the Base of an Exponential Expression
The General Case of Changing Base of an Exponential Expression
Changing the Base of a Logarithm
The General Case of Changing Base of a Logarithm
Some Exercises on Changing Base

Exercise 1: $\log_b a = \dfrac{1}{\log_a b}$

Exercise 2: $\log_7 29$

Exercise 3: $\log_{(a^p)}x = \dfrac{1}{p}\log_a x$

Exercise 4: $\log_7 5 = \dfrac{\log_{10}5}{\log_{10}7}$

6. Graphs of Logarithms

Summary   *6. Graphs of Logarithms*

Movie   *6. Graphs of Logarithms*

The Purpose of This Movie
Exploring a Logarithmic Graph
The Graph $y = \log_a x$ when $0 < a < 1$

Overview of the Library on Functions, Graphs, and Trigonometry

The Introduction to Analytic Geometry Group

1. Graphing with a Computer Algebra System

Summary   *1. Graphing with a Computer Algebra System*

Worksheet *1. Graphing with a Computer Algebra System*

Movie   *1. Graphing with a Computer Algebra System*

Cartesian Coordinates in the Plane
Rectangular 2D Graphs
Exercises on Rectangular 2D Graphs

Exercise 1: $y = x^3 - x^2$

Exercise 2: $y = x^2(1-x)^2$

Exercise 3: $y = \left(\sqrt[3]{x}\right)^2\sqrt[3]{6-x}$

Exercise 4: $y = x\sin\dfrac{1}{x}$

Exercise 5: $y = \sin x + \sin\dfrac{6}{5}x$

More General Graphs of Equations: Implicit 2D Graphs
Exercises on Implicit 2D Graphs

Exercise 1: $x^2 y - y^2 + xy^3 = 5$

Exercise 2: $x^2 - 3xy + 4y^2 - 5x^4 - 20y^4 - 14x^6 + 16y^6 + 15xy^3 + 15x^3 y - 12xy^5 + 42x^5 y - 25x^2 y^2 + 72x^2 y^4 - 51x^3 y^3 - 39x^4 y^2 = 0$

Exercise 3: $4x^4 - 5y^2 - 5x^2 + 4y^4 + 17x^2 y^2 + 1 = 0$

Exercise 4: $6x^2 - 3xy + 9y^2 - 9x^4 - 24y^4 + 4x^6 + 16y^6 + 15xy^3 + 15x^3y - 12xy^5 - 12x^5y - 42x^2y^2 + 72x^2y^4 - 51x^3y^3 + 33x^4y^2 - 1 = 0$

Exercise 5: $x^3 + y^3 - 3xy = 0$
Exercise 6: $x^3 + y^3 + 3x^2y^2 = 0$
Exercise 7: $x^5 + y^5 - 3x^2y = 0$
Exercise 8: $xy(x^2 - y^2) = x^2 + y^2$
Exercise 9: $x^2y^2(x^2 - y^2) = x^2 + y^2$
Exercise 10: $x^2y(x^2 + y^2) = 100(x^2 - y^2)$
Exercise 11: $x^2 - y^2 = 3$

2. The Distance Formula

Summary ***2. The Distance Formula***
Worksheet ***2. The Distance Formula***

Movie ***2. The Distance Formula***

Statement of the Distance Formula
Length of a Horizontal Line Segment
Length of a Vertical Line Segment
Length of a General Line Segment
Exercises on The Distance Formula
Exercise 1: AB
    Exercise 1a: $A = (1, 2)$ and $B = (9, 17)$
    Exercise 1b: $A = (-2, 0)$ and $B = (3, 12)$
    Exercise 1c: $A = (-3, 5)$ and $B = (-3, -9)$
Exercise 2: $29^2 = (x - (-3))^2 + (21 - 1)^2$
Exercise 3: $2x + y - 2 = 0$
Exercise 4
Exercise 5
Exercise 6
Exercise 7: Showing that a given triangle is a right triangle.
Exercise 8: Showing that a given triangle is a right triangle.
Exercise 9: Showing that a given triangle is a right triangle.
Exercise 10: $AP + BP = 8$

3. Slope of a Line Segment

Summary ***3. Slope of a Line Segment***
Worksheet ***3. Slope of a Line Segment***

Movie ***3. Slope of a Line Segment***

Definition of Slope of a Line Segment
The Order of the Points Is Unimportant
The Concept of Slope Is not Defined for Vertical Line Segments
Horizontal Line Segments Have Zero Slopes
Rising Line Segments Have Positive Slopes
Falling Line Segments Have Negative Slopes
Some Elementary Exercises on Slope
Exercise 1: $\text{slope}(AB)$
    Exercise 1a: $A = (-1, 2)$ and $B = (2, 8)$
    Exercise 1b: $A = (-1, 2)$ and $B = (2, -6)$
    Exercise 1c: $A = (-1, 2)$ and $B = (3, 2)$
    Exercise 1d: $A = (-1, 2)$ and $B = (-1, 5)$
Exercise 2: $\text{slope}(AB) = \text{slope}(CD)$
Exercise 3: AB
    Exercise 3a: $\text{slope}(AB) = -\dfrac{4}{7}$
    Exercise 3b: $\text{slope}(AB) = \dfrac{4}{7}$
Exercise 4: $\text{slope}(AP) = 2$
Exercise 5: $A = (2, -3)$, $B = (4, 3)$ and $P = (x, y)$
    Exercise 5a: $\text{slope}(AB) = \text{slope}(AP)$
    Exercise 5b: $\text{slope}(AB) = \text{slope}(BP)$

4. Properties of Slope

Summary ***4. Properties of Slope***
Worksheet ***4. Properties of Slope***

Movie ***4. Properties of Slope***

When Line Segments Are Parallel to One Another

5. Equation of a Line

Summary   5. Equation of a Line

Movie   5. Equation of a Line

6. Exercises on Lines

Summary   6. Exercises on Lines
Worksheet 6. Exercises on Lines

Movie   6. Exercises on Lines

Exercise 9b: $OG = \dfrac{1}{2}GH$

7. Circle Graphs

Summary   7. Circle Graphs

Movie   7. Circle Graphs

Worksheet 7. Circle Graphs

Introduction to Circles
An Example of a Circle
General Form of the Equation of a Circle
The Domain Intervals of a Circle
Some Exercises on Circles
Exercise 1: $(x-4)^2 + (y+3)^2 = 7$
Exercise 2: $(x-2)^2 + (y+5)^2 = 0$
Exercise 3: $(x-2)^2 + (y+5)^2 = -9$
Exercise 4: $(x-2)^2 + (y-3)^2 = 16$
Exercise 5: $(x+2)^2 + (y-3)^2 = 25$
Exercise 6: $(x-2)^2 + (y+1)^2 = 13$
Exercise 7: $x^2 + (y-6)^2 = 0$
Exercise 8: $x^2 + (y-6)^2 = -4$
Exercise 9: $(x-2)^2 + (y-1)^2 \le 9$
Exercise 10: $(x-2)^2 + (y-1)^2 < 9$
Exercise 11: $(x-2)^2 + (y-1)^2 \ge 9$
Exercise 12: $(x-2)^2 + (y-4)^2 = 26$ and $(x+1)^2 + (y-2)^2 = 13$
The Upper and Lower Halves of a Circle
More Exercises on Circles
Exercise 13: $(x-2)^2 + (y-5)^2 = 4$
Exercise 14: $\left\{ (x,y) \ | \ 0 \le x \le 4 \text{ and } \frac{3}{4}x \le y \le \sqrt{25 - x^2} \right\}$
Exercise 15: $\left\{ (x,y) \ | \ 0 \le x \le 4 \text{ and } 4 \le x^2 + y^2 \le 25 \text{ and } y \ge \frac{3}{4}x \right\}$
Exercise 16: $(x-1)^2 + (y-2)^2 = 10$
The Equation $x^2 + y^2 + ax + by + c = 0$
Exercise 1: $x^2 + y^2 + 4x - 10y - 20 = 0$
Exercise 2: $x^2 + y^2 - 3x + 8y = 0$
Exercise 3: $x^2 + y^2 - 6x + 4y - 12 = 0$
Exercise 3a: $x^2 - 6x + y^2 + 4y = 12$
Exercise 3b: $(x-3)^2 + (y+2)^2 = -4 + 4 + 9$
Exercise 3c: $(x-3)^2 + (y+2)^2 = -9 + 4 + 9$
Exercise 3d: $(x-3)^2 + (y+2)^2 = -12 + 4 + 9$
Exercise 3e: $(x-3)^2 + (y+2)^2 = -17 + 4 + 9$

The Course in Trigonometry Group

1. Angles in Geometry and in Trigonometry

Summary   1. Angles in Geometry and in Trigonometry
Worksheet 1. Angles in Geometry and in Trigonometry

Movie   1. Angles in Geometry and in Trigonometry

Angles in Geometry
Angles as Rotations
Initial Line and Terminal Line of an Angle
Angles Coterminal to Each Other
Angles Drawn in Standard Position
Radian Measure of An Angle
The Role of The Number π in Meaurement of Area of a Disk
The Role of The Number π in Meaurement of Length of a Circle
Definition of a Radian
The Area of a Circular Sector
The Length of a Circular Arc
Exercises on Circular Sectors
Exercise 1: $\pi(16)^2 \left(\dfrac{1}{6} \right)$
Exercise 2: $\dfrac{63360}{40\pi}$

Exercise 3: $x = \dfrac{6}{2\pi}$

Exercise 4: $\dfrac{2\pi r}{15}$

2. Introduction to Trigonometry

Summary 2. Introduction to Trigonometry
Worksheet 2. Introduction to Trigonometry

Movie 2. Introduction to Trigonometry

The Names of the Trigonometric Functions
Right Triangles: The Historical Origins of Trigonometry
The Special Angle $45°$
The Special Angles $60°$ and $30°$

Exercises on Acute Angle Trigonometry

Exercise 1

Exercise 1a: $\dfrac{u}{6} = \cos 40°$

Exercise 1b: $\dfrac{6}{u} = \sin 40°$

Exercise 1c: $\dfrac{u}{6} = \sin 40°$

Exercise 1d: $\dfrac{6}{u} = \tan 40°$

Exercise 1e: $\dfrac{u-1}{u} = \sin 40°$

Exercise 1f: $\dfrac{u-1}{2u-5} = \tan 40°$

Exercise 2: $\triangle ABD$

Exercise 3: $\dfrac{6}{u} = \tan 43°$

Exercise 4: $\angle ABC$

The Transition to General Trigonometry

Drawing an Acute Angle in Standard Position
Defining the Trigonometric Functions at Any Angle
When the Terminal Line Has Length 1
Animated Demonstration of the Definition of cos and sin
The Trigonometric Functions Applied to Numbers
Example to Illustrate the Cosine of a Number
Another Example to Illustrate the Cosine of a Number
The Signs of the Trigonometric Functions

Some Exercises on the Trigonometric Functions

Exercise 1: Angles Coterminal to $0°$
Exercise 2: Angles Coterminal to $90°$
Exercise 3: Angles Coterminal to $180°$
Exercise 4: Angles Coterminal to $270°$
Exercise 5: line OP where $P = (2, -3)$
Exercise 6: line OP. Given that P lies in Quadrant II, that $P = (x, 8)$
Exercise 7: line OP of length 7. Given that $P = (2, y)$
Exercise 8: line OP of length 13
Exercise 9: $\sin \theta = \dfrac{24}{25}$

Exercise 10: $0 < \theta < 180°$ and $\tan \theta = -3$
The Pythagorean Relationship
Notation for the Square of a Trigonometric Function

Revisiting Exercises 8, 9, and 10 on the Trigonometric Functions

Revisiting Exercise 8: $\cos \theta = -\dfrac{12}{13}$

Revisiting Exercise 9: $\cos^2 \theta + \sin^2 \theta = 1$
Revisiting Exercise 10: $1 + \tan^2 \theta = \sec^2 \theta$

The Graphs of the Trigonometrìc Functions

The Graphs of $\sin$ and $\cos$
The Graph of $\tan$
The Graph of $\cot$
The Graph of $\sec$
The Graph of $\csc$

Application of Trigonometry to Musical Chords

The Distinction Between Noise and Music
Combining Two Musical Notes that are an Octave Apart
Playing a Discord
Combining Two Musical Notes that are a Fifth Apart
Combining Two Musical Notes that are a Fourth Apart
Combining Two Musical Notes that are a Major Third Apart
Combining Two Musical Notes that are a Minor Third Apart

When Do Two Musical Notes Harmonize?
An Optional Extra for Registered Users of Scientific Notebook

3. Trigonometry in a General Triangle

Summary **3.** *Trigonometry in a General Triangle*
Worksheet **3.** *Trigonometry in a General Triangle*

Movie **3.** *Trigonometry in a General Triangle*

The Objective of this Movie
Placing a Triangle into a Coordinate System
The Law of Cosines
Introducing Area of a Triangle
Area of a Parallelogram
Area of a Triangle Using Base and Height
Area of a Triangle Using Sides and Angles
The Law of Sines

Exercises on Finding the Sides of a Triangle
Exercise 1: $\angle A = 60°$
Exercise 2: $\angle A = 40°$ and $\angle C = 57°$
Exercise 3: $\triangle ABC$
Exercise 4: $\angle B = 45°$
Exercise 5: $\angle B = 55°$
Exercise 6: $\cos \angle C$ and $\sin \angle C$
Exercise 7: $a^2 = b^2 + c^2 - 2bc \cos \angle A$
Exercise 8: $\angle B = 45°$ and $\angle C = 75°$
Exercise 9: $b^2(a + c) = a^3 + c^3$
Exercise 10: $\cos \angle C$
Exercise 11: $5^2 = 7^2 + b^2 - (2)(7)b \cos 33°$
Exercise 12: $\angle A = 90°$
Exercise 13: $\dfrac{a}{\sin 90°} = \dfrac{b}{\sin \angle B} = \dfrac{c}{\sin \angle C}$
Exercise 14: $x_1 x_2 + y_1 y_2 = \sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2} \cos \theta$

Finding The Angles of A Triangle
The Cosine of an Angle in a Triangle
    The Angle $\arccos 0$
    The Angle $\arccos \dfrac{1}{2}$
    The Angle $\arccos\left(-\dfrac{1}{2}\right)$
    The Angles $\arccos\left(\dfrac{1}{\sqrt{2}}\right)$ and $\arccos\left(-\dfrac{1}{\sqrt{2}}\right)$
    The Angles $\arccos\left(\dfrac{\sqrt{3}}{2}\right)$ and $\arccos\left(-\dfrac{\sqrt{3}}{2}\right)$
    The Angles $\arccos(.37)$ and $\arccos(-.37)$
Finding an Angle in a Triangle when its Cosine Value is Known
Example of an Angle with a Known Cosine Value
The Sine of an Angle in a Triangle
    The Angle $\arcsin 1$
    The Angle $\arcsin \dfrac{1}{2}$
    The Angle $\arcsin \dfrac{1}{\sqrt{2}}$
Finding an Angle in a Triangle when its Sine Value is Known
Exercises on Finding the Angles of a Triangle
Exercise 1: $\angle B$
Exercise 2: $\dfrac{7}{\sin 55°} = \dfrac{6}{\sin \angle B}$
Exercise 3: $\dfrac{5}{\sin 55°} = \dfrac{7}{\sin \angle B}$
Exercise 4: $c^2 = 3^2 + 17^2 - (2)(3)(17) \cos 20°$
Exercise 5: $\dfrac{10}{\sin 60°} = \dfrac{12}{\sin \angle B}$
Exercise 6: $\dfrac{12}{\sin \angle B} = \dfrac{6\sqrt{3}}{\sin 60°}$
Exercise 7: $\dfrac{12}{\sin \angle B} = \dfrac{11}{\sin 60°}$
Exercise 8: $(AB)^2 = (OA)^2 + (OB)^2 - \cos \angle AOB$
Exercise 9: $(AB)^2 = (OA)^2 + (OB)^2 - 2(OA)(OB) \cos \angle AOB$
Special Topic: **Area of a Triangle in Terms of Coordinates**

4. Analytic Trigonometry in One Variable

Summary   **4. Analytic Trigonometry in One Variable**
Worksheet   **4. Analytic Trigonometry in One Variable**

Movie   **4. Analytic Trigonometry in One Variable**

Preliminary Remarks
A Word of Warning
A Story About Bertrand Russell
Summary of the Relationships Between the Trig Functions
The Method of Reducing to cos and sin Only

Exercises on Proving Trigonometric Identities

Exercise 1: $\dfrac{\sec x - \cos x}{\tan x} = \dfrac{\tan x}{\sec x}$

Exercise 2: $\dfrac{1 + \cos\theta}{\sin\theta} + \dfrac{\sin\theta}{1 + \cos\theta} = 2\csc\theta$

Exercise 3: $\tan^2\theta - \sin^2\theta = \tan^2\theta\sin^2\theta$

Exercise 4: $1 - \tan^4\theta = 2\sec^2\theta - \sec^4\theta$

Exercise 5: $\dfrac{\cos\theta}{\tan\theta(1 - \sin\theta)} = \csc\theta + 1$

Exercise 6: $\dfrac{\sec x + \csc x}{\sec x - \csc x} = \dfrac{\sin x + \cos x}{\sin x - \cos x}$

Exercise 7: $\dfrac{1 + \sin x}{1 - \sin x} - \dfrac{1 - \sin x}{1 + \sin x} = 4\tan x\sec x$

Exercise 8: $\dfrac{1 + \csc\beta}{\sec\beta} - \cot\beta = \cos\beta$

Exercise 9: $\dfrac{\cos x\cot x}{\cot x - \cos x} = \dfrac{\cot x + \cos x}{\cos x\cot x}$

Exercise 10: $(\sec u - \tan u)(\csc u + 1) = \cot u$

Exercise 11: $(\csc\alpha - \cot\alpha)^2 = \dfrac{1 - \cos\alpha}{1 + \cos\alpha}$

Exercise 12: $\dfrac{\cot\theta - \tan\theta}{\sin\theta + \cos\theta} = \csc\theta - \sec\theta$

Exercise 13: $\csc^4 x - \cot^4 x = \cot^2 x + \csc^2 x$

Exercise 14: $\cos^4 x + \sin^2 x = \sin^4 x + \cos^2 x$

Exercise 15: $\dfrac{1 + \tan^2\theta}{\sin\theta} + \dfrac{1 + \cot^2\theta}{\cos\theta} = \dfrac{\sin\theta + \cos\theta}{\sin^2\theta\cos^2\theta}$

Exercise 16: $\dfrac{\cos\beta}{1 - \sin\beta} = \sec\beta + \tan\beta$

Exercise 17: $\dfrac{\tan^2 x}{\sec x + 1} = \dfrac{1 - \cos x}{\cos x}$

Exercise 18: $\csc^2\alpha - \csc\alpha\cot\alpha = \dfrac{1}{1 + \cos\alpha}$

Exercise 19: $\sin^4 x + 2\sin^2 x\cos^2 x + \cos^4 x = 1$

Exercise 20: $(\sec x + \tan x)^2 = \dfrac{1 + \sin x}{1 - \sin x}$

Exercise 21: $\dfrac{\sin\theta + \cos\theta}{\tan^2\theta - 1} = \dfrac{\cos^2\theta}{\sin\theta - \cos\theta}$

Exercise 22: $\dfrac{\cos^3 x - \sin^3 x}{\cos x - \sin x} = 1 + \sin x\cos x$

Exercise 23: $\dfrac{\cos^6 x - \sin^6 x}{\cos^2 x - \sin^2 x} = \cos^2 x + \sin^4 x$

Exercise 24: $\sin^6 x + \cos^6 x = 1 - 3\sin^2 x\cos^2 x$

Exercise 25: $\dfrac{\sin^6 u - \cos^6 u}{\sin^4 u - \cos^4 u} = \sin^2 u + \cos^4 u$

Exercise 26: $\dfrac{\sin\theta}{1 - \cos\theta} = \dfrac{1 + \cos\theta}{\sin\theta}$

Exercise 27: $\sqrt{\dfrac{1 - \sin\theta}{1 + \sin\theta}} = \dfrac{1 - \sin\theta}{|\cos\theta|}$

Exercise 28: $\sqrt{\dfrac{1 - \cos\theta}{1 + \cos\theta}} = \dfrac{1 - \cos\theta}{|\sin\theta|}$

Exercise 29: $\sqrt{\dfrac{1 - \sin x}{1 + \sin x}} = \dfrac{|\cos x|}{1 + \sin x}$

Exercise 30: $\dfrac{\cot x - \tan x}{\sin x\cos x} = \csc^2 x - \sec^2 x$

Exercise 31: $\dfrac{\tan x}{1 - \cot x} + \dfrac{\cot x}{1 - \tan x} = 1 + \sec x \csc x$

Exercise 32: $(1 - \tan^2 \alpha)^2 = \sec^4 \alpha - 4 \tan^2 \alpha$

Exercise 33: $\dfrac{\cos x}{1 - \tan x} + \dfrac{\sin x}{1 - \cot x} = \cos x + \sin x$

Exercise 34: $\log_3 \left(\dfrac{1 + \cos x}{1 - \cos x} \right) = 2 \log_3 \left(\dfrac{|\sin x|}{1 - \cos x} \right)$

Exercise 35: $\log_2(1 + \cos \theta) - \log_2(1 - \cos \theta) = 2 \log_2 |\csc \theta + \cot \theta|$

5. Analytic Trigonometry in Several Variables

Summary   5. Analytic Trigonometry in Several Variables

Worksheet 5. Analytic Trigonometry in Several Variables

Movie   5. Analytic Trigonometry in Several Variables

Introduction

The Package of Four

Exercises on the Package of Four

Exercise 1: $\cos 75°$ and $\sin 75°$

Exercise 2: $\sin 285°$

Exercise 3: $\sin 15°$

Exercise 4: $\tan 105°$

Exercise 5: $\cos 255°$

Exercise 6: $\tan 1095°$

Exercise 7: $\cos(90° - \theta) = \sin \theta$

Exercise 8: $\sin(90° - \theta) = \cos \theta$

Exercise 9: $\tan(90° - \theta) = \cot \theta$

Exercise 10: $\cos(-\theta) = \cos \theta$

Exercise 11: $\sin(-\theta) = -\sin \theta$

Exercise 12: $\sin(90° + \theta)$

Exercise 13: $\tan(\alpha + \beta) = \dfrac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$

Exercise 14: $\tan(\alpha - \beta) = \dfrac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$

Exercise 15: $\cot(\alpha + \beta) = \dfrac{\cot \alpha \cot \beta - 1}{\cot \alpha + \cot \beta}$

Exercise 16: $\sin \alpha = \dfrac{1}{3}$ and $\cos \beta = \dfrac{2}{5}$

    Exercise 16a: $\cos \alpha$ and $\sin \beta$.

    Exercise 16b: $\sin(\alpha + \beta)$

    Exercise 16c: $\cos(\alpha - \beta)$

Exercise 17: $\sin(\alpha - \beta) \cos \beta + \sin \beta \cos(\alpha - \beta) = \sin \alpha$

Exercise 18: $\sin(\alpha - \beta + \gamma) \cos \beta \cos \gamma + \cos(\alpha - \beta + \gamma) \sin \beta \cos \gamma - \sin \gamma \cos(\alpha + \gamma) = \sin \alpha$

Double-Angle Identities

Half-Angle Identities

Triple-Angle Identities

Exercises on Multiple-Angle Identities

Exercise 1: $\cos 2\theta$

Exercise 2: $\sin 2\theta$

Exercise 3: $\cos^2 \theta = \dfrac{1}{2}(1 + \cos 2\theta)$

Exercise 4: $\cos^4 \theta - \sin^4 \theta = \cos 2\theta$

Exercise 5: $\cos \left(22 \dfrac{1}{2}° \right)$

Exercise 6: $\cos \dfrac{\theta}{2}$

Exercise 7: $\cos 4\theta = 8 \cos^4 \theta - 8 \cos^2 \theta + 1$

Exercise 8: $\cos^4 \theta + \sin^4 \theta = 1 - \dfrac{1}{2} \sin^2(2\theta)$

Exercise 9: $\cos^8 \theta - \sin^8 \theta = (\cos 2\theta) \left(1 - \dfrac{1}{2} \sin^2(2\theta) \right)$

Exercise 10: $\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$

Exercise 11: $\sin 3\theta = 3 \sin \theta - 4 \sin^3$

Exercise 12: $\cos \theta = \dfrac{1 - t^2}{1 + t^2}$ and $\sin \theta = \dfrac{2t}{1 + t^2}$

Exercise 13: $\sin 18$

Factorization Identities

Exercises on Factorization Identities

Exercise 1: $\sin 8\theta + \sin 2\theta = 2 \sin 5\theta \cos 3\theta$

Exercise 2: $\dfrac{\sin 5\theta + \sin 3\theta}{\sin 5\theta - \sin 3\theta} = \tan 4\theta \cot \theta$

Exercise 3: $\dfrac{\sin\theta + \sin 2\theta + \sin 4\theta + \sin 5\theta}{\cos\theta + \cos 2\theta + \cos 4\theta + \cos 5\theta} = \tan 3\theta$

Exercise 4: $\cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta = 4\cos\theta\cos 2\theta\cos 4\theta$

Exercise 5: $\sin 2x + \sin 4x + \sin 6x = 4\sin 3x\cos 2x\cos x$

6. Deriving the Package Identities

Summary   6. Deriving the Package Identities

Movie   6. Deriving the Package Identities

Our Objective in this Movie
The Principle of Addition of Angles
    First Example on Addition of Angles
    Second Example on Addition of Angles
    Third Example on Addition of Angles
Proof of the Identity first package identity
The Rest of the Package
Using Only the First Identity in the Package to expand $\cos(-\theta)$, $\sin(-\theta)$, $\cos(90° - \theta)$, and $\sin(90° - \theta)$
Expanding $\cos(-\theta)$
Expanding $\cos(90° - \theta)$
Expanding $\sin(90° - \theta)$
Expanding $\sin(-\theta)$
Expanding $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
Expanding $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$
Expanding $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$

7. Inverse Trigonometric Functions

Summary   7. Inverse Trigonometric Functions
Worksheet 7. Inverse Trigonometric Functions

Movie   7. Inverse Trigonometric Functions

The Idea of the Function $\arccos$
Some Examples to Illustrate the Function $\arccos$
The Angle $\arccos 0$
The Angle $\arccos \dfrac{1}{2}$
The Angle $\arccos\left(-\dfrac{1}{2}\right)$
The AngleS $\arccos\left(\dfrac{1}{\sqrt{2}}\right)$ and $\arccos\left(-\dfrac{1}{\sqrt{2}}\right)$
The AngleS $\arccos\left(\dfrac{\sqrt{3}}{2}\right)$ and $\arccos\left(-\dfrac{\sqrt{3}}{2}\right)$
The AngleS $\arccos(.37)$ and $\arccos(-.37)$
Some Basic Facts about $\arccos$
The Function $\arcsin$
The Function $\arcsin$ as We Saw it in the Movie on Triangles
Introducing the Function $\arcsin$ General Case
Some Examples to Illustrate the Function $\arcsin$
The Angles $\arcsin 1$ and $\arcsin(-1)$
The Angles $\arcsin\left(\dfrac{1}{2}\right)$ and $\arcsin\left(-\dfrac{1}{2}\right)$
The Angles $\arcsin\left(\dfrac{1}{\sqrt{2}}\right)$ and $\arcsin\left(-\dfrac{1}{\sqrt{2}}\right)$
Some Basic Facts about $\arcsin$
A Relationship Between $\arcsin$ and $\arccos$
The Function $\arctan$
Some Examples to Illustrate the Function $\arctan$
The Angle $\arctan 0$
The Angles $\arctan 1$ and $\arctan(-1)$
The Angles $\arctan\sqrt{3}$ and $\arctan\left(-\sqrt{3}\right)$
The Angles $\arctan\dfrac{1}{\sqrt{3}}$ and $\arctan\left(-\dfrac{1}{\sqrt{3}}\right)$
The Angles $\arctan 1000000000$ and $\arctan(-1000000000)$
Some Basic Facts about $\arctan$
An Identity Involving $\arctan$

The Function arcsec

Some Examples to Illustrate the Function arcsec

The Angles $\arcsec 1$ and $\arcsec(-1)$

The Angles $\arcsec 2$ and $\arcsec(-2)$

The Angles $\arcsec \sqrt{2}$ and $\arcsec\left(-\sqrt{2}\right)$

Some Basic Facts about arcsec

Exercises on Inverse Trigonometric Functions

Exercise 1: $\cos(2\arcsin u) + \cos(2\arccos u) = 0$

Exercise 2: $\arccos(\cos\theta) = \theta$

Exercise 3: $\cos(3\arccos u) = 4u^3 - 3u$

Exercise 4: $\sin(4\arccos u) = 4u(2u^2 - 1)\sqrt{1 - u^2}$

Exercise 5: $\tan(2\arctan u)$

Exercise 6: $\arctan u + \arctan\left(\dfrac{1}{u}\right) = -\dfrac{\pi}{2}$

Exercise 7: $\arcsin(-u) = -\arcsin u$

Exercise 8: $\arccos(-u) = \pi - \arccos u$

Exercise 9: $\arctan\left(\dfrac{1 - \cos\theta}{\sin\theta}\right) + \arctan(\cot\theta) = \dfrac{\pi - \theta}{2}$

8. Solution of Trigonometric Equations

Summary   8. Solution of Trigonometric Equations
Worksheet 8. Solution of Trigonometric Equations

Movie   8. Solution of Trigonometric Equations

Solving the Cosine Equation: Geometric Approach
Motivating the Solution of the Cosine Equation
Solution of the Cosine Equation: Geometric Approach
(Optional) Solving the Cosine Equation: Algebraic Approach
Solving the Sine Equation: Geometric Approach
Motivating the Solution of the Sine Equation
Solution of the Sine Equation: Geometric Approach
(Optional) Solving the Sine Equation: Algebraic Approach
Solving the Tangent Equation

Exercises on Trigonometric Equations

Exercise 1: $\cos 3\theta = \dfrac{1}{2}$

Exercise 2: $\sqrt{2}\sin 2\theta = -1$

Exercise 3: $\sqrt{3}\tan 3\theta = 1$

Exercise 4: $4\cos^2 5\theta = 3$

Exercise 5: $6\cos^2 6\theta + 7\sin 6\theta - 8 = 0$

Exercise 6: $\cos 2\theta = \cos(100° + \theta)$

Exercise 7: $\sin 3\theta = \sin\theta$

Exercise 8: $\cos 5\theta = \sin(34° + 3\theta)$

Exercise 9: $\sec^2\theta + \tan\theta - 3 = 0$

Exercise 10: $\tan^2 3\theta + 3\sec 3\theta - 9 = 0$

Exercise 11: $\cos 6\theta - \cos 2\theta = 0$

Exercise 12: $\cos 7\theta + \cos 3\theta = \cos 2\theta$

Exercise 13: $\cos 3\theta - \cos 5\theta = \sin 4\theta$

Exercise 14: $\cos 3\theta + \cos 5\theta = \cos\theta$

Exercise 15: $\sqrt{3}\cos\theta + \sin\theta - \sqrt{2} = 0$

The Relations and Functions Group

1. Introduction to Relations

Summary   1. Introduction to Relations
Worksheet 1. Introduction to Relations

Movie   1. Introduction to Relations

Some Examples of Relations
The Relation "Is a Brother of"
The Relation "Same Color as"
Associating a Color to Each Bead in a Set of Beads
Associating to Each Color, the Beads that Match it.
The Relation "Is a Factor of"
Calorie Intake and Body Weight
A Circle Relation
A Disk Relation

A Relation Given by a More General Inequality

Exercises on Relations

Exercise 1: $[0, 1]$ when $x = y$

Exercise 2: $[-1, 1]$ when $x = y$

Exercise 3: $[0, 1]$ and $x \leq y$

Exercise 4: $[0, 1]$ and $x \leq \dfrac{1}{2}$ and $y \geq \dfrac{1}{2}$

Exercise 5: $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ when x is a multiple of y.

Exercise 6: $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ when $x - y$ is an odd number

Exercise 7: $(x - 2)^2 + (y - 3)^2 = 9$

Exercise 8: $(x - 2)^2 + (y - 3)^2 \leq 9$

Exercise 9: $(x - 2)^2 + (y - 3)^2 \geq 9$

Exercise 10: $10 \leq (x - 2)^2 + (y - 3)^2 \leq 25$

Exercise 11: $10 \leq (x - 2)^2 + (y - 3)^2 \leq 25$ and $y \geq 2x - 6$

2. Introduction to Functions

Summary   2. Introduction to Functions
Worksheet 2. Introduction to Functions

Movie   1. Introduction to Functions

Intuitive Definition of a Function

Some Examples of Functions

Example 1: $\left\{ (-1, 2), (0, 2), (1, 1), \left(2, \sqrt{3} \right), (3, -1), (4, 1), (5, 3) \right\}$

Example 2: Associating a Color to Each Bead in a Set of Beads

Example 3: $(x, x^2 - 2x)$ for which $0 \leq x \leq 3$

A Return to The Examples of Relations

Associating to Each Color, the Beads that Match it

Calorie Intake and Body Weight

The Relation "Is a Brother of"

The Relation "Same Color as"

A Return to the Relation "Factor of"

A Return to the Circle Relation

A Return to the Disk Relation

Function Notation and Domain of a Function

Illustrating Function Notation with the Bead Color Example

Distinguishing between a Function and Its Values

Some Further Examples of Functions

Naming a Person

A Quadratic Function

A Restricted Quadratic Function

A Piecewise Defined Function

Exercises on Functions

Exercise 1: $f(x) = x^2$ for every real number x

Exercise 2: $f(x) = x^2$ for every number $x \geq 2$

Exercise 3: $f(x) = x^2$

Exercise 4: $f(x) = \begin{cases} \sqrt{x - 2} & \text{if } x > 5 \\ \sqrt{1 - x} & \text{if } x \leq 1 \end{cases}$

Exercise 5: $f(x) = \dfrac{\sqrt{9 - x^2}}{1 - \sqrt{x - 1}}$

Function Images of a Set

Function from a Set A to a Set B

Some Examples to Illustrate the Notion $\mathbf{f : A \to B}$

Example 1: $f(x) = x^2$ for every $x \in [-2, 3]$

Example 2: $f(x) = \begin{cases} -2 & \text{if } x \leq -3 \\ x^2 & \text{if } -3 < x \leq 5 \\ 0 & \text{if } x > 5 \end{cases}$

The Range of a Function

Defining the Image of a Set Under a Function

Examples of Function Images

Example 1: $f(x) = x^2$ for every real number x

Example 2: $f(x) = \dfrac{x}{1 + x^2}$

Exercises on Images and Ranges

Exercise 1

Exercise 1a: $f\left[\left[-\sqrt{2},3\right]\right]$
Exercise 1b: $f\left[\left(-\sqrt{2},3\right)\right]$
Exercise 1c: $f\left[\left(-\sqrt{2},3\right]\right]$
Exercise 1d: $f\left[\left[-\sqrt{2},3\right)\right]$

Exercise 2

Exercise 2a: $f\left[\left[\sqrt{2},3\right]\right]$
Exercise 2b: $f\left[\left(\sqrt{2},3\right)\right]$
Exercise 2c: $f\left[\left[\sqrt{2},3\right)\right]$
Exercise 2d: $f\left[\left(\sqrt{2},3\right]\right]$

Exercise 3: $f(x) = \begin{cases} x & \text{if } 0 \le x < 1 \\ 3-x & \text{if } 1 \le x \le 2 \end{cases}$

Exercise 4: $f(x) = x^2 - 6x + 11$
Exercise 5: $f(x) = (x^2 - 6x + 5)^2$
Exercise 6: $f(x) = (x^2 - 6x + 5)^2$ for $1 \le x \le 5$
Exercise 7: $f(x) = \dfrac{x-2}{1-2x}$
Exercise 8: $f(x) = \dfrac{3x-2}{x+1}$
Exercise 9: $f(x) = \dfrac{x^2+1}{x^2+x+1}$

3. Some Elementary Topics on Functions

Summary   **3. Some Elementary Topics on Functions**
Worksheet **3. Some Elementary Topics on Functions**

Movie   **3. Some Elementary Topics on Functions**

Vertical Shifting of Graphs
Example to Illustrate Verical Shifting
Horizontal Shifting of Graphs
An Example to Motivate the Idea of Horizontal Shifting
Some Further Shifting of the Graph $y = x^2$
The Principle of Horizontal Shifting
Animating the Graph $y = (x-c)^2$
Animating the Graph $y = (x-c)^2 - 3(x-c)^2$
Combining Functions
Arithmetical Combinations of Functions
Examples on Arithmetical Combinations of Functions
Example 1: $f(x) = 3x - 2$ and $g(x) = \sqrt{x^2+1}$
Example 2: $f(x) = 3x - 2$ and $g(x) = x^2 - 9$
Example 3: $f(x) = 3x - 2$ and $g(x) = x^2 - 9$ for every number $x \in [-2,2]$
Composition of Functions
Some Examples of Compositions
Example 1: $(g \circ f)(x) = g(f(x)) = g(x^2) = x^2 + 1$
Example 2: $(f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = 1 - (\sqrt{x})^2 = 1 - x$
Example 3: $(g \circ f)(x) = g(f(x)) = g(1 - x^2) = \sqrt{1 - x^2}$
Example 4: $(f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = 1 - (\sqrt{x})^2 = 1 - x$
Exercises on Compositions
Exercise 1: $(g \circ f)(2)$ and $(f \circ g)(2)$
Exercise 2: $(g \circ f)(2)$
Exercise 3: $f \circ g$ and $g \circ f$
Exercise 4
   Exercise 4a: $f(x) = \sqrt{x}$ for all $x \ge 0$ and $g(x) = x^2$ for all numbers x
   Exercise 4b: $f(x) = \sqrt{x}$ for all $x \ge 0$ and $g(x) = x^2$ for all $x \ge 0$
   Exercise 4c: $f(x) = \dfrac{x-2}{1-2x}$ for all $x \ne \dfrac{1}{2}$ and $g(x) = \dfrac{x+2}{1+2x}$ for all $x \ne -\dfrac{1}{2}$
   Exercise 4d: $f(x) = 1 + 2x$ for all numbers x and $g(x) = 3 - x$ for all numbers x
Exercise 5: $h(x) = \sqrt[3]{1 + \sqrt{1 + x^2}}$
Exercise 6: $(f \circ (g \circ h))(x) = f((g \circ h)(x))$
Exercise 7
   Exercise 7a: $c = \dfrac{a+b}{1+ab}$
   Exercise 7b: $c = \dfrac{a-b}{1-ab}$

Exercise 7c: $f_a(x) = \dfrac{x-a}{1-ax}$

4. Inverse Function of a Given Function

Summary   4. Inverse Function of a Given Function
Worksheet 4. Inverse Function of a Given Function

Movie   4. Inverse Function of a Given Function

One-One Functions
Definition of a One-One Function (Injective Function)
Examples on One-One Functions
Example 1: $f(x) = x^3$ for every number x
Example 2: $f(x) = x^2$ for every number $x \geq 0$
Example 3: $f(x) = x^2$ for every number x
Example 4: $f(x) = \dfrac{x}{x+1}$
The Inverse Function of a One-One Function
Definition of the Inverse Function of a Given Function
How to Find the Inverse Function of a Given Function
Examples of Inverse Functions of Given Functions
Example 1: $f(x) = 3x - 2$ for every number x
Example 2: $f(x) = x^2$ for every $x \in [0, \infty)$
Example 3: $f(x) = \dfrac{x}{x+1}$ for every number $x \geq 0$
Example 4: $f(x) = 2^x$ for every number x
Example 5: $f(x) = \dfrac{3x-1}{3-x}$ for $-1 < x < 1$
  Some Preliminary Inequalities
  Now we begin Example 5
Example 6: $f(x) = \dfrac{3x^3-1}{3-x^3}$ for $-1 < x < 1$
Exercises on Inverse Functions of Given Function
Exercise 1: $f(x) = x + 3$
Exercise 2: $f(x) = 2x - 6$
Exercise 3: $f(x) = x^3$
Exercise 4: $f(x) = \sqrt[3]{x}$
Exercise 5: $f(x) = x + \dfrac{1}{x}$
Exercise 6: $f(x) = x + \dfrac{1}{x+1}$
Exercise 7: $f(x) = \dfrac{x}{\sqrt{x+1}}$
Monotone Functions
Increasing Functions
Decreasing Functions
Definition of a Monotone Function
Monotone Functions and One-One Functions