Virtual Calculus Tutor

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Document: 2.2 Introduction to the Limit Concept

Movie: 2.2 Introduction to the Limit Concept

2.2.1 Motivating the Idea of a Limit

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Example 4: $f(x) = \dfrac{x^2 - 9}{x - 3}$ for $x \neq 3$

Example 5: $f(x) = x + 3$ for all x

Example 6: $f(x) = \begin{cases} x + 3 & \text{if} \quad x \neq 3 \\ 4 & \text{if} \quad x = 3 \end{cases}$

Example 7: $f(x) = \begin{cases} x - 1 & \text{if} \quad x < 3 \\ 5 - x & \text{if} \quad x > 3 \end{cases}$

Example 8: $(x) = \begin{cases} x - 1 & \text{if} \quad x < 3 \\ 2 - x & \text{if} \quad x > 3 \end{cases}$

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Movie: 2.3 Properties of Limits

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2.3.2 The Arithmetical Rules

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Document: 2.4 Trigonometric Limits

Movie: 2.4 Trigonometric Limits

2.4.1 Radian Measure and Area of a Circular Sector

The Number π

Radian Measure of an Angle

Area of a Circular Sector

Evaluating Trigonometric Functions at a Number

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2.4.3 Obtaining the Trigonometric Limits

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2.4.4 Exercises on the Trigonometric Limits

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Exercise 9: $\lim\limits_{\theta \to 0} \frac{\tan\theta - \sin\theta}{\theta^3}$

Exercise 10: $\lim\limits_{\theta \to 0} \frac{1 - \sqrt[3]{\cos\theta}}{\theta^2}$

Exercise 11: $\lim\limits_{\theta \to 0} \frac{\sqrt[3]{\cos 3\theta} - \sqrt[3]{\cos 5\theta}}{\theta^2}$

Exercise 12: $\lim\limits_{x \to 0+} \sin \frac{1}{x}$

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Document: 2.5 Continuity

Movie: 2.5 Continuity

2.5.1 Introducing the Concept of Continuity

Review of the Intuitive Definition of a Limit

Definition of Continuity of a Function f at a Number x

2.5.2 Some Examples to Illustrate the Idea of a of Continous Function

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Example 3: $f(t) = t + 3$ for $t \neq 3$

Example 4: $f(t) = \frac{t^2 - 9}{t - 3}$ when $t - 3 \neq 0$

Example 5: $f(t) = \begin{cases} \frac{t^2 - 9}{t - 3} & \text{if } t \neq 3 \\ 6 & \text{if } t = 3 \end{cases}$

Example 6: $f(t) = \begin{cases} \frac{t^2 - 9}{t - 3} & \text{if } t \neq 3 \\ 2 & \text{if } t = 3 \end{cases}$

Example 7: $f(t) = \begin{cases} \frac{t^2 - 9}{t - 3} & \text{if } t < 3 \\ 6 & \text{if } t = 3 \end{cases}$

Example 8: $f(t) = \begin{cases} t + 3 & \text{if } t < 3 \\ 6 & \text{if } t = 3 \\ 2 - t & \text{if } t > 3 \end{cases}$

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The Intermediate Value Property
Maxima and Minina of Continuous Functions
The Theorem on Existence of Maxima and Minima of Continuous Functions

2.5.4 Some Examples of Functions that Fail to Have a Maximum or a Minimum

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The Effect of a Discontinuity

2.5.5 Exercises on the Properties of Continuous Functions

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Exercise 2: $f(x) = x^2$ for $-3 < x < 3$
Exercise 3: $f(x) = \begin{cases} x & \text{if } 0 < x < 2 \\ x-2 & \text{if } 2 \leq x \leq 4 \end{cases}$
Exercise 4: $f(x) = |x^2 - 4|$ for $0 \leq x \leq 5/2$
Exercise 5: $f(x) = \begin{cases} x & \text{if } 0 \leq x < 1 \\ 1 + 4x - x^2 & \text{if } 1 \leq x \leq 4 \end{cases}$
Exercise 6: Existence of a solution of $5\sqrt[3]{x} + \sqrt{9-x} = 6$

Overview of Chapter 3: Derivatives

Document: 3.1 Introduction to Derivatives

Movie: 3.1 Introduction to Derivatives

3.1.1 Definition of a Derivative

Motivating the Definition Using Slopes
Definition of the Derivative of a Function
Alternative Form of the Definition of a Derivative

3.1.2 Some Examples of Derivatives

Example 1: Derivative of a constant
Example 2: $f(x) = mx + b$ for all x
Example 3: $f(x) = x^2$ for all x, find $f'(3)$
Example 4: $f(x) = x^2$ for all x, find $f'(x)$
Example 5: $f(x) = x^3$ for all x, find $f'(x)$
Example 6: $f(x) = x^7$ for all x, find $f'(x)$

3.1.3 The Power Rule

Introducing the Power Rule
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The Power Rule for the Case $p = 5/6$
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3.1.4 Derivatives of Polynomials

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3.1.5 The Leibniz Notation for Derivatives

Motivating the Leibniz Notation for Derivatives
Introducing the Leibniz Notation for Derivatives
The Power Rule in Leibniz Notation
Derivative of a Polynomial in Leibniz Notation

3.1.6 Exercises on Derivatives

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Exercise 2: $\frac{d}{dx}\frac{5}{\sqrt[4]{x^7}}$

Exercise 3: $y = 8x^3 - 6x - 1$ tangent line problem

Exercise 4: $y = \frac{1}{\sqrt{x}}$ tangent line problem

Exercise 5: Tangent from $(-2,-21)$ to $y = x^2$

Exercise 6: Tangent from $(-3,1)$ to $y = \frac{1}{x}$

Exercise 7: $f(x) = |x-3|$ no derivative at 3

Exercise 8: $\lim_{x\to 0} x \sin\frac{1}{x} = 0$

Exercise 9: $x^2 \sin\frac{1}{x}$ derivative at 0?

Document: 3.2 Elementary Facts About Derivatives

Movie: 3.2 Elementary Facts About Derivatives

3.2.1 The Rules for Differentiation

The Sum Rule
- Stating the Sum Rule
- Explaining the Sum Rule

The Difference Rule
- Stating the Difference Rule
- Explaining the Difference Rule

The Constant Multiple Rule
- Stating the Constant Multiple Rule
- Explaining the Constant Multiple Rule

The Product Rule
- Stating the Product Rule
- A Needed Fact About Limits
- Explaining the Product Rule

The Quotient Rule
- Stating the Quotient Rule
- Explaining the Quotient Rule
- An Optional Deeper Comment About the Proof of the Quotient Rule

3.2.2 Exercises on the Rules for Differentiation

Exercise 1: $f(x) = x + \frac{1}{x}$ for $x \neq 0$

Exercise 2: Tangent line from $(4,4)$ to $y = x + \frac{1}{x}$

Exercise 3: $\frac{d}{dx}\frac{x}{1+x^2}$

Exercise 4: Horizontal tangents to $y = \frac{x^2}{1+x^4}$

Exercise 5: $y = \frac{x}{x^2+4}$ tangent line problem

Exercise 6: $f(x) = (x-3)^2 g(x)$ tangent line problem

Exercise 7: $\frac{d}{dx} f(x)g(x)h(x)$ extended product rule

Exercise 8: $\frac{d}{dx}\left(f(x)\right)^2 = 2f(x)f'(x)$

Exercise 9: Horizontal tangents to $y = (2-3x)^5(5+2x)^4$

3.2.3 Higher Order Derivatives

3.2.4 Exercises on Higher Order Derivatives

Exercise 1: $f(x) = x^7$ for each x, work out $f^{(n)}(x)$

Exercise 2: $f(x) = \sqrt{x}$ for each $x > 0$, work out $f^{(n)}(x)$

Exercise 3: $f(x) = \frac{1}{1+x^2}$ find $f''(x)$

Exercise 4: Expand $(1+x)^7$ using derivatives

Exercise 5: Expand $(1+x)^p$ using derivatives

Document: 3.3 Derivatives of the Trigonometric Functions

Movie: 3.3 Derivatives of the Trigonometric Functions

3.3.1 Derivatives of the Functions sin and cos

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The Derivative of cos

3.3.2 Derivatives of the Other Trigonometric Functions

The Derivative of tan
Finding the Derivative of tan Directly from the Definition
The Derivative of cot
Finding the Derivative of cot Directly from the Definition
The Derivative of sec
Finding the Derivative of sec Directly from the Definition
The Derivative of csc
Finding the Derivative of csc Directly from the Definition
Summary of the Trigonometric Derivatives

3.3.3 Exercises on Derivatives of the Trigonometric Functions

Exercise 1: $\frac{d}{dx}\frac{\sin x}{x}$

Exercise 2: $\frac{d}{dx}x^2 \sin x \cos x$

Exercise 3: $\frac{d}{dx}\frac{x \sin x}{1+x^2}$

Exercise 4: Horizontal tangents to $y = 2\cos^2 x + 2\cos x - 1$

Exercise 5: $\frac{d}{dx}\left((f(x)-\sin x)^2 + (g(x)-\cos x)^2\right)$

Document: 3.4 Derivative of a Composition

Movie: 3.4 Derivative of a Composition

3.4.1 Composition of Functions

3.4.2 Some Examples of Compositions

Example 1: $f(x) = x^2$ for every number x and $g(u) = 3 + 5u$ for every number u

Example 2: $f(x) = 1 + x^2$ for every number x and $g(u) = u^{100}$ for every number u

Example 3: $f(x) = 2^x$ for every number x and $g(u) = \log_2 u$ for $u > 0$

Example 4: $f(x) = \frac{x-2}{1-2x}$ whenever $x \neq \frac{1}{2}$ and $g(u) = \frac{u-3}{1-3u}$ for $u \neq \frac{1}{3}$

3.4.3 Statement of the Composition Rule

3.4.4 Some Examples to Illustrate the Composition Rule

Example 1: $\frac{d}{dx}(1+x^2)^{100}$

Example 2: $\frac{d}{dx}\sin(1+x^2)$

Example 3: $\frac{d}{dx}\sqrt{\sin x}$

3.4.5 Motivating the Composition Rule

3.4.6 Using Leibniz Notation in the Composition Rule

3.4.7 A Return to the Earlier Examples on the Composition

Example 1: $\frac{d}{dx}(1+x^2)^{100}$

Example 2: $\frac{d}{dx}\sin(1+x^2)$

Example 3: $\frac{d}{dx}\sqrt{\sin x}$

3.4.8 Some Assorted Exercises on Derivatives

Exercise 1: $\frac{d}{dx}\sqrt{\sin(1+x^2)}$

Exercise 2: $\frac{d}{dx}(\sin x + \cos x)^{100}$

Exercise 3: $\frac{d}{dx}\sqrt{\sin\sqrt{x}}$

Exercise 4: $\frac{d}{dx}\left(\sin x + x\cos(x^3)\right)^{100}$

Exercise 5: $\frac{d}{dx}\frac{\sqrt{\sin(x^3)}}{\sqrt[3]{\cos(x^2)}}$

Exercise 6: Tangent to $y = \tan x$ at $x = \pi/4$

Exercise 7: Tangent to $y = \sqrt{13-x^2}$ at $(5,1)$

Exercise 8: Finding the Angle Between Two Graphs

Exercise 9: Angle of intersection of $y = \sin x$ and $y = \cos x$

Note on the Final Two Exercises

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Exercise 11: The Whispering Gallery Problem

Document: 3.5 Inverse Functions

Movie: 3.5 Inverse Functions

3.5.1 Domain and Range of a Function

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Example 2 on Domain and Range
Example 3 on Domain and Range
Example 4 on Domain and Range

3.5.2 Inverse Function of a One-One Function

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3.6.5 Some Exercises on Derivatives of Exponential and Logarithmic Functions

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Exercise 2: $\frac{d}{dx} \log(5x)$

Exercise 3: $\frac{d}{dx} \log 5 = 0$

Exercise 4: $f(x) = \log(1 + x^2)$

Exercise 5: $\frac{d}{dx} \log|\sin x|$

Exercise 6: $\frac{d}{dx} \log|\sec x|$

Exercise 7: $\frac{d}{dx} \log|\sec x + \tan x|$

Exercise 8: $\frac{d}{dx} \log|\csc x + \cot x|$

Exercise 9: $\frac{d}{dx} (1 + x^2)^{\sin x}$

Exercise 10: $\frac{d}{dx} \log_{(1+x^2)}(1 + x^2 + 2x^4)$

Exercise 11: $\lim_{x\to 0}(1 + x)^{1/x}$

Exercise 12: $\lim_{u\to\infty}(1 + \frac{1}{u})^u$

Document: 3.7 Inverse Trigonometric Functions

Movie: 3.7 Inverse Trigonometric Functions

3.7.1 The Function arccos

3.7.2 Some Examples to Illustrate the Function arccos

The Number $\arccos 0$
The Number $\arccos \frac{1}{2}$
The Number $\arccos\left(-\frac{1}{2}\right)$
The Numbers $\arccos \frac{1}{\sqrt{2}}$ and $\arccos\left(-\frac{1}{\sqrt{2}}\right)$
The Numbers $\arccos\left(\frac{\sqrt{3}}{2}\right)$ and $\arccos\left(-\frac{\sqrt{3}}{2}\right)$
The Numbers $\arccos(.37)$ and $\arccos(-.37)$

3.7.3 Some Properties of the Function arccos

Working Out $\cos(\arccos x)$, $\sin(\arccos x)$, and $\tan(\arccos x)$
The Derivative of the Function arccos
The Graph of the Function arccos

3.7.4 The Function arcsin

3.7.5 Some Examples to Illustrate the Function arcsin

The Numbers $\arcsin 1$ and $\arcsin(-1)$

The Numbers $\arcsin\frac{1}{2}$ and $\arcsin\left(-\frac{1}{2}\right)$

The Numbers $\arcsin\frac{1}{\sqrt{2}}$ and $\arcsin\left(-\frac{1}{\sqrt{2}}\right)$

3.7.6 Some Properties of the Function arcsin

Working Out $\sin(\arcsin x)$, $\cos(\arcsin x)$, and $\tan(\arcsin x)$
The Derivative of the Function arcsin
The Graph of the Function arcsin

3.7.7 The Function arctan

3.7.8 Some Examples to Illustrate the Function arctan

The Number $\arctan 0$
The Numbers $\arctan 1$ and $\arctan(-1)$
The Numbers $\arctan\sqrt{3}$ and $\arctan\left(-\sqrt{3}\right)$
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The Limits of arctan at ∞ and at $-\infty$

3.7.9 Some Properties of the Function arctan

Working out $\tan(\arctan x)$, $\sec(\arctan x)$, and $\sin(\arctan x)$
The Identity $\arctan x + \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2}$ for $x > 0$
Derivative of the Function arctan
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3.7.10 The Function arcsec

3.7.11 Some Examples to Illustrate the Function arcsec

The Numbers $\operatorname{arcsec} 1$ and $\operatorname{arcsec}(-1)$
The Numbers $\operatorname{arcsec} 2$ and $\operatorname{arcsec}(-2)$
The Numbers $\operatorname{arcsec}\sqrt{2}$ and $\operatorname{arcsec}\left(-\sqrt{2}\right)$

3.7.12 Some Properties of the Function arcsec

Working Out $\sec(\operatorname{arcsec} x)$, $\tan(\operatorname{arcsec} x)$, and $\sin(\operatorname{arcsec} x)$
The Derivative of the Function arcsec
The Graph of the Function arcsec

3.7.13 Exercises on Inverse Trigonometric Functions

Exercise 1: $\arctan\left(\sqrt{2} - 1\right) = \frac{\pi}{8}$
Exercise 2: $\arctan\left(2 - \sqrt{3}\right) = \frac{\pi}{12}$
Exercise 3: $\cos(2\arcsin u) + \cos(2\arccos u) = 0$
Exercise 4: $\arccos(\cos\theta) = \theta$?
Exercise 5: $\cos(3\arccos x) = 4x^3 - 3x$
Exercise 6: $\sin(4\arccos x) = 4x(2x^2 - 1)\sqrt{1 - x^2}$
Exercise 7: $\tan(2\arctan x)$ defined?
Exercise 8: $\arctan x + \arctan\left(\frac{1}{x}\right) = -\frac{\pi}{2}$ for $x < 0$
Exercise 9: $\arcsin(-x) = -\arcsin x$
Exercise 10: $\arccos(-x) = \pi - \arccos x$
Exercise 11: $\arctan\left(\frac{1 - \cos\theta}{\sin\theta}\right) + \arctan(\cot\theta) = \frac{\pi - \theta}{2}$

Document: 3.8 Implicit Functions

Movie: 3.8 Implicit Functions

3.8.1 Implicit 2D Graphs

Example 1: $x^2 + y^2 = 25$
Example 2: $x^2 y - y^2 + xy^3 = 5$
Example 3: $(x^2 + y^2)^2 = x^2 - y^2$
Example 4: $x^3 + y^3 - 3xy = 0$
Example 5: $x^5 + y^5 - 3x^2 y = 0$
Example 6: $x\sin(x^2 + y^2) + y = 0$

3.8.2 The Implicit Function Theorem

3.8.3 Some Exercises on Implicit Functions

Exercise 1: Tangent to $x^2 + y^2 = 25$ at $(3, 4)$
Exercise 2: Slope of $x^2 y - y^2 + xy^3 = 5$ at a general point (x, y)
Exercise 3: Tangent to $x^2 y - y^2 + xy^3 = 5$ at $(2, 1)$
Exercise 4: Slope of $(x^2 + y^2)^2 = x^2 - y^2$ at a general point (x, y)
Exercise 5: Horizontal and vertical tangents to $x^3 + y^3 - 3xy = 0$
Exercise 6: Horizontal and vertical tangents to $x^5 + y^5 - 3x^2 y = 0$
Exercise 7: Slope of $x\sin(x^2 + y^2) + y = 0$ at a general point (x, y)

Document: 3.9 Hyperbolic Functions

Movie: 3.9 Hyperbolic Functions

3.9.1 Introduction to Hyperbolic Functions

Some Preliminary Comments
The Definitions of the Hyperbolic Functions

3.9.2 Arithmetical Properties of the Hyperbolic Functions

Behaviour of the Hyperbolic Functions at 0
"Pythagorean Identities" for the Hyperbolic Functions
Replacing x by $-x$ in the Hyperbolic Functions
Hyperbolic Function Values at a Sum or Difference
Analogues for the Hyperbolic Functions of the Trigonometric Double and Triple Angle Identities

3.9.3 Derivatives of the Hyperbolic Functions

The Equation $\frac{d}{dx}\sinh x = \cosh x$
The Equation $\frac{d}{dx}\cosh x = \sinh x$
The Equation $\frac{d}{dx}\tanh x = \operatorname{sech}^2 x$
The Equation $\frac{d}{dx}\operatorname{sech} x = -\operatorname{sech} x \tanh x$

3.9.4 Inverse Functions of the Hyperbolic Functions

The Function $\operatorname{arcsinh}$
Finding $\frac{d}{dx}\operatorname{arcsinh} x$
The Function $\operatorname{arccosh}$
Finding $\frac{d}{dx}\operatorname{arccosh} x$
The Function $\operatorname{arctanh}$
Finding $\frac{d}{dx}\operatorname{arctanh} x$

The Function arcsech

Finding $\frac{d}{dx} \operatorname{arcsech} x$

3.9.5 Some Derivatives that Involve the Hyperbolic Functions

Example 1: $\frac{d}{dx} \arctan(\sinh x)$

Example 2: $\frac{d}{dx} \arctan(e^x)$

Example 3: $\frac{d}{dx} \arcsin(\operatorname{sech} x)$

Example 4: $\frac{d}{dx} \log(\cosh x)$

Example 5: $\frac{d}{dx} \log(\sinh x)$

Example 6: $\frac{d}{dx} \operatorname{arcsec}(\cosh x)$

Example 7: $\frac{d}{dx} \arccos(\operatorname{sech} x)$

Example 8: $\frac{d}{dx} \operatorname{arccosh}(\sec x)$

Example 9: $\frac{d}{dx} \operatorname{arctanh}(\sin x)$

Overview of Chapter 4: Applications of the Derivative

Document: 4.1 Monotone Functions

Movie: 4.1 Monotone Functions

4.1.1 The Graph of a Function with a Positive Derivative

The Positive Derivative Principle
Looking at the Positive Derivative Principle Intuitively
A Note of Caution

4.1.2 Increasing and Decreasing Functions

Strictly Increasing Functions
Increasing Functions
Strictly Decreasing Functions
Decreasing Functions
Monotone Functions

4.1.3 More General Version of the Positive Derivative Principle

4.1.4 Exercises on Monotone Functions

Exercise 1: $f(x) = x^2 - 4x - 5$ for all x

Exercise 2: $f(x) = x^3 - 3x^2$ for all x

Exercise 3: $f(x) = f(x) = |x^2 - 4x - 5|$ for all x

Exercise 4 $f(x) = |x^3 - 3x^2|$ for all x

Exercise 5: $f(x) = \left(\frac{\log x}{x}\right)^2$ for $x > 0$

4.1.5 An Application of the Positive Derivative Principle

The Inequality $e^x > 1$ when $x > 0$

The Inequality $e^x > 1 + x$ when $x > 0$

The Inequality $e^x > 1 + x + \frac{x^2}{2}$ when $x > 0$

The Inequality $e^x > 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}$ when $x > 0$
The General Case $e^x > 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$ when $x > 0$

4.1.6 Working out Some Important Limits

The Limit $\lim_{x\to\infty} \frac{e^x}{x}$
The Limit $\lim_{x\to\infty} \frac{e^x}{x^5}$
The Limit $\lim_{x\to\infty} \frac{e^x}{x^n}$
The Limit $\lim_{x\to\infty} \frac{\log x}{x}$
The Limit $\lim_{x\to\infty} \frac{(\log x)^{1000000}}{x}$
The Limit $\lim_{x\to 0+} x \log x$
The Limit $\lim_{x\to 0+} x(\log x)^{1000000}$

Document: 4.2 Drawing Graphs of Functions

Movie: *4.2 Drawing Graphs of Functions*

4.2.1 Maxima and Minima

Definition of Maxima and Minima
Definition of Local Maxima and Minima

4.2.2 Fermat's Theorem

Statement of Fermat's Theorem
- Part 1: Positive derivative not at the right endpoint
- Part 2: Negative derivative not at the right endpoint
- Part 3: Positive derivative not at the left endpoint
- Part 4: Negative derivative not at the left endpoint
- Part 5: Conclusion

Using Fermat's Theorem
Critical Numbers of a Function

4.2.3 Some Examples to Illustrate Fermat's Theorem

Example 1: $f(x) = x^3$ for $-2 \le x \le 2$
Example 2: $f(x) = x^3$ for $-2 \le x < 2$
Example 3: $f(x) = \begin{cases} \frac{x-1}{2} & \text{if } 0 \le x \le 3 \\ 2x-5 & \text{if } 3 \le x \le 4 \end{cases}$

4.2.4 Exercises on Graphs of Functions

Exercise 1: $f(x) = x^2 - 4x - 5$ for $-2 \le x \le 6$
Exercise 2: $f(x) = x^2 - 4x - 5$ for $3 \le x \le 6$
Exercise 3: $f(x) = |x^2 - 4x - 5|$ for $-2 \le x \le 6$
Exercise 4: $f(x) = x^3 - 3x^2$ for $-1 \le x \le 4$
Exercise 5: $f(x) = \frac{x^2}{1+x^2}$ for all x
Exercise 6: $f(x) = xe^{-x}$ for $x \ge -1$
Exercise 7: $f(x) = xe^{-x^2}$ for all x
Exercise 8: $f(x) = x^2 e^{-x^2}$ for all x
Exercise 9: $f(x) = 3\sin^4 x - 2\sin^3 x$ for $0 \le x \le 2\pi$
Exercise 10: $f(x) = x(\log x)^2$ for $0 < x \le 2$
Exercise 11: $f(x) = x^{2/3}(6-x)^{1/3}$ for $-1 \le x \le 7$

4.2.5 Concavity of Graphs

The Graph of a Function with a Positive Second Derivative
The Graph of a Function with a Negative Second Derivative
Points of Inflection

4.2.6 Exercises on Concavity

Exercise 1: $f(x) = x^3 - 3x^2$ for all x
Exercise 2: $f(x) = \frac{x^2}{1+x^2}$ for all x
Exercise 3: $f(x) = xe^{-x}$ for all x
Exercise 4: $f(x) = xe^{-x^2}$ for all x
Exercise 5: $f(x) = x^2 e^{-x^2}$ for all x
Exercise 6: $f(x) = \log(1 + x^2)$ for all x
Exercise 7: $f(x) = (\log x)^2$ for $x > 0$
Exercise 8: $f(x) = x(\log x)^2$ for $x > 0$
Exercise 9: $f(x) = x(\log(x^2))^2 - 3x\log(x^2)$ for $x \neq 0$
Exercise 10: $f(x) = x^{2/3}(6-x)^{1/3}$ for $-1 \leq x \leq 7$
Exercise 11: $f(x) = \frac{x\log x}{1+x^2}$ for $x > 0$
Exercise 12: Theoretical

Document: 4.3 Applied Maxima and Minima

Movie: 4.3 Applied Maxima and Minima

4.3.1 Elementary Exercises on Applied Maxima and Minima

Exercise 1: The Chicken Coop Problem
Exercise 2: The Box Problem
Exercise 3: The Cylindrical Can Problem
Exercise 4: The Rectangle in a Semicircle Problem
Exercise 5: The Isosceles Triangle in a Parabola Problem
Exercise 6: The Isosceles Triangle in a Circle Problem
Exercise 7: The Cone in a Hemisphere Problem
Exercise 8: The Triangle and Semicircle Problem
Exercise 9: The Road and Field Problem (Special Case)
Exercise 10: The Dimmer Switch Problem
Exercise 11: An Electric Circuit Problem

4.3.2 The General Road and Field Problem (and Deriving Snell's Law)

The Narrow Road Version of the Road and Field Problem
The Wide Road Version of the Road and Field Problem
The Road and Field Problem and the Laws of Refraction
Comparing the Wide Road Problem with the Narrow Road Problem

4.3.3 Making a Quadrilateral of Maximum Area

Maximizing the Area of a Quadrilateral with Given Sides
The Three Sticks Problem

4.3.4 The Ice Cream Problem: Maximum Minimum Problems About Cones

Background Information About Cones
Maximizing the Volume of a Cone with a Given Slant Height
Minimizing the Slant Height of a Cone with a Given Volume
Maximizing the Volume of a Cone with Given Surface Area
Filling the Cone with Ice Cream

4.3.5 Introducing The Soapbox Car Problem (See Section 8.4 for the full discussion.)

Document: 4.4 Antiderivatives (Indefinite Integrals)

Movie: *4.4 Antiderivatives (Indefinite Integrals)*

4.4.1 Antiderivative of a Function

4.4.2 Some Examples of Antiderivatives

Example 1: Antiderivative with respect x of $6x$
Example 2: Another antiderivative with respect x of $6x$
Example 3: Antiderivative with respect x of $\cos x$
Example 4: Antiderivative with respect x of $\frac{1}{x}$ when $x > 0$
Example 5: Antiderivative with respect x of $\frac{1}{x}$ when $x < 0$
Example 6: Antiderivative with respect x of $\frac{1}{x}$ when $x \neq 0$
Example 7: Antiderivative with respect x of x^p when $p \neq -1$

4.4.3 The Key Fact About Antiderivatives

Statement of the Key Fact
Finding all Possible Antiderivatives of a Given Function

4.4.4 Some Examples of General Antiderivatives

Example 1: $\int x dx = \frac{x^2}{2} + c$

Example 2: $\int x^p dx = \frac{x^{p+1}}{p+1} + c$

Example 3: $\int \frac{1}{x} dx = \log|x| + c$

Example 4: $\int \cos x dx = \sin x + c$

Example 5: $\int \sin x dx = -\cos x + c$

Example 6: $\int \sec^2 x dx = \tan x + c$

Example 7: $\int \sec x \tan x dx = \sec x + c$

Example 8: $\int \tan x dx = \log|\sec x| + c$

Example 9: $\int \cot x dx = \log|\sin x| + c$

Example 10: $\int \sec x dx = \log|\sec x + \tan x| + c$

Example 11: $\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + c$

Example 12: $\int \frac{1}{1+x^2} dx = \arctan x + c$

Example 13: $\int \frac{1}{x\sqrt{x^2-1}} dx = \operatorname{arcsec} x + c$

4.4.5 Changing Variable to Find an Antiderivative

Motivating the Change of Variable Method: Example 1
Motivating the Change of Variable Method: Example 2
Motivating the Change of Variable Method: Example 3
Motivating the Change of Variable Method: Example 4
Motivating the Change of Variable Method: Example 5
Motivating the Change of Variable Method: Example 6
Motivating the Change of Variable Method: Example 7
Introducing the Change of Variable Method
Applying The Change of Variable Method

4.4.6 Some Exercises on Changing Variable

Exercise 1: $\int \sqrt{1+x^2}\, 2x dx$

Exercise 2: $\int \frac{4x+3}{\sqrt{2x^2+3x+7}} dx$

Exercise 3: $\int \frac{x}{1+x^2} dx$

Exercise 4: $\int \cos^4 x \sin x dx$

Exercise 5: $\int \sqrt{\tan x}\, \sec^2 x dx$

Exercise 6: $\int \frac{\cos(\log x)}{x} dx$

Exercise 7: $\int e^x \sin(3e^x) dx$

Exercise 8: $\int (\log \sin x)^2 \cot x dx$

Exercise 9: $\int x\sqrt{x+3}\, dx$

Exercise 10: $\int \sqrt{\sin x}\, \cos x dx$

Exercise 11: $\int \sqrt{\sin x}\, \cos^3 x dx$

Exercise 12: $\int \sqrt{\sin x}\, \cos^5 x dx$

Exercise 13: $\int \sqrt{\cos x}\, \sin^5 x dx$

Exercise 14: $\int \sec^6 x \sqrt{\tan x}\, dx$

Exercise 15: $\int \sec^3 x \tan^5 x dx$

Exercise 16: $\int (1+x) dx$ (two ways)

Exercise 17: $\int \sin 2\theta d\theta$ (two ways)

Exercise 18: $\int \frac{1}{1-x^2} dx$

Exercise 19: $\int \sec x dx$

Exercise 20: $\int \csc x dx$

4.4.7 Antiderivatives that Involve Hyperbolic Functions

Exercise 1: $\int \cosh x dx = \sinh x + c$

Exercise 2: $\int \sinh x dx = \cosh x + c$

Exercise 3: $\int \operatorname{sech} x dx = 2 \arctan(e^x) + c$

Exercise 4: $\int \tanh x dx = \log \cosh x + c$

Exercise 5: $\int \sqrt[3]{\tanh x}\, \operatorname{sech}^2 x dx$

Exercise 6: $\int \frac{1}{\sqrt{x^2+1}} dx = \operatorname{arcsinh} x + c$

Exercise 7: $\int \frac{\operatorname{arcsinh} x}{\sqrt{x^2+1}} dx$

Exercise 8: $\int \frac{\cos x}{\sqrt{1+\sin^2 x}} dx$

Exercise 9: $\int \frac{\sqrt{\operatorname{arccosh} x}}{\sqrt{x^2 - 1}} dx$

Exercise 10: $\int \frac{1}{1 - x^2} dx = \operatorname{arctanh} x + c$

Exercise 11: $\int \frac{1}{x\sqrt{1 - x^2}} dx = -\operatorname{arcsech} x + c$

Document: 4.5 Rates of Change

Movie: 4.5 Rates of Change

4.5.1 Interpreting the Derivative as a Rate of Change

4.5.2 Some Exercises on Derivatives as Rates of Change

Exercise 1. Inflating a Balloon: Part 1
Exercise 2. Inflating a Balloon: Part 2
Exercise 3. A Leaking Cone: Part 1
Exercise 4. A Leaking Cone: Part 2
Exercise 5. Water Evaporating from a Cone
Exercise 6. Growth of a Bacterial Colony
Exercise 7. Growth of Money in a Bank Account
Exercise 8. Radioactive Decay

Document: 4.6 Motion of a Particle in a Straight Line

Movie: 4.6 Motion of a Particle in a Straight Line

4.6.1 The Position Function of a Moving Particle

4.6.2 Examples to Illustrate Position Functions

Example 1: $f(t) = t^2$ for $-1 \le t \le 1$
Example 2: $f(t) = t^4$ for $-1 \le t \le 1$
Example 3: $f(t) = t^2$ for $t \ge 0$
Example 4: $f(t) = \sin t$ for $t \ge 0$

4.6.3 Velocity, Speed, and Acceleration of a Particle

4.6.4 Some Exercises on Velocity, Speed, and Acceleration

Exercise 1: $f(t) = t^2$ at each time t
Exercise 2: $f(t) = \sin t$ at each time t in the interval $[0, 6\pi]$
Exercise 3: $f'(t) = 5t$ for each time t
Exercise 4: $f''(t) = 20$ for every t

4.6.5 Expressing Velocity and Acceleration in Terms of Postion

An Example to Illustrate Velocity and Acceleration at a Point x
A Formula for Velocity in Terms of Position
Returning to the Example
A Formula for Acceleration in Terms of Position
Returning, Once Again, to the Example

4.6.6 Newton's Law

Introducing the Concept of Mass
Introducing the Concept of Force
Introduction to Newton's Law

Overview of Chapter 5: The Mean Value Theorem and its Applications

Exercise 6: When the inequality $|f(t) - f(x)| \leq |t - x|^2$ holds
Exercise 7: A condition for two functions to be $\sin$ and $\cos$
Exercise 8: Derivatives have an intermediate value property
Exercise 9: A two function version of Exercise 8

Document: 5.2 Approximating a Function with Polynomials

Movie: *5.2 Approximating a Function with Polynomials*

5.2.1 Introduction to Polynomials

Definition of a Polynomial
Expanding $(1 + x)^8$: Motivating the Binomial Theorem
The Binomial Theorem

5.2.2 The Coefficients of a General Polynomial

Special Notation for Higher Derivatives of a Function
Finding the Coefficients of a Given Polynomial
The Degree of a Polynomial
Recentering the Terms of a Polynomial

5.2.3 Taylor Polynomials of a Function

Definition of The Taylor Polynomials

5.2.4 Some Examples of Taylor Polynomials

Example 1: $f(x) = 2 - 4x + 3x^2 + 7x^3 + 5x^4$ for each x
Example 2: $f(x) = \frac{1}{1 + x^2}$ for each x
Example 3: $f(x) = \frac{1}{1 + x^2}$, Taylor polynomials centered at 1
Example 4: Using a computer algebra system to find Taylor polynomials
Example 5: Another application of a computer algebra system

5.2.5 Finding The Remainder Term

Introducing the Remainder Term of a Taylor Polynomial
A Quick Review of Rolle's Theorem
A Version of Rolle's Theorem for the Second Derivative
A Version of Rolle's Theorem for the Third Derviative
A Version of Rolle's Theorem for the Fourth Derivative

Motivating the Higher Derivative Form of the Mean Value Theorem: A Mean Value Theorem for the Fourth Derivative

The Higher Deriverative Form of the Mean Value Theorem (Sometimes Called the Taylor Mean Value Theorem)

5.2.6 Some Applications of the Taylor Mean Value Theorem

Finding an Approximation to e
The Number e is Irrational
Finding an Approximation to e^3
Finding an approximation to $\log(\frac{3}{2})$
Finding an approximation to $\log(\frac{1}{2})$
Finding An Approximation to $\cos 1$
The Number $\cos 1$ Is Irrational

Document: 5.3 Indeterminate Forms

Movie: *5.3 Indeterminate Forms*

5.3.1 Introduction to Indeterminate Forms

5.3.2 Some Examples to Illustrate Indeterminate Forms

Example 1: $\lim_{x\to 0} \frac{3x}{x} = 3$

Example 2: $\lim_{x\to 0} \frac{\sin x}{x} = 1$

Example 3: $\lim_{x\to 0+} x(\log x) = 0$

Example 4: $\lim_{x\to \infty} \frac{(\log x)^3}{x} = 0$

Example 5: $\lim_{x\to 0} (1+2x)^{1/x} = e^2$

Example 6: $\lim_{x\to \infty} \left(\sqrt{x^2+3x+1} - \sqrt{x^2-2x+7}\right) = \frac{5}{2}$

5.3.3 L'Hôpital's rule

Introducing L'Hôpital's rule
More Careful Statement of L'Hôpital's rule
Some Remarks About L'Hôpital's rule
- The Rule Works for One-Sided and Two-Sided Limits
- The Limit May Be Finite or Infinite
- The Case in Which $\lim_{x\to a} g(x) = \infty$
- A Brief History of L'Hôpital's rule

A Special Case of L'Hôpital's Rule
- Example 1 Showing Use of the Special Case of L'Hôpital's Rule
- Example 2 Showing Use of the Special Case of L'Hôpital's Rule
- Proof of the Special Case of L'Hôpital's Rule

5.3.4 Exercises on Indeterminate Forms

Exercise 1: $\lim_{x\to \infty} \frac{3x-7}{2x+5}$

Exercise 2: $\lim_{x\to 0} \frac{e^x-1}{x}$

Exercise 3: $\lim_{x\to 0} \frac{e^x \sin 5x - \sin 3x}{x}$

Exercise 4: $\lim_{x\to 0} \left(\frac{\tan x - x}{x - \sin x}\right)$

Exercise 5: $\lim_{x\to 0} \left(\frac{x - \sin x}{x^3}\right)$

Exercise 6: $\lim_{x\to 0+} x \log x$

Exercise 7: $\lim_{x\to 1} \frac{\log x}{x-1}$

Exercise 8: $\lim_{x\to \infty} \frac{\log x}{x}$

Exercise 9: $\lim_{x\to \infty} \frac{(\log x)^2}{x}$

Exercise 10: $\lim_{x\to \infty} \frac{(\log x)^p}{x} = 0$

Exercise 11: $\lim_{x\to \infty} \left(\log(3x+2) - \log(2x-5)\right)$

Exercise 12: $\lim_{x\to \infty} \frac{\log(x+2)}{\log(x-5)}$

Exercise 13: $\lim_{x\to \infty} \left(\frac{(\log(3x+2))^2 - (\log(2x-5))^2}{\log x}\right)$

Exercise 14: $\lim_{x\to \infty} \frac{\exp\left(\sqrt{\log x}\right)}{x}$

Exercise 15: $\lim_{x\to \pi/2} (\sin x)^{\tan x}$

Exercise 16: $\lim_{x\to \infty} x^{(\log x)/x}$

Exercise 17: $\lim_{x\to 0}\left((1+px)^{1/x}\right)$

Exercise 18: $\lim_{x\to 0}\dfrac{e-(1+x)^{1/x}}{x}$

Exercise 19: $\lim_{x\to\infty}\left((\log(x+1))^2-(\log x)^2\right)$

Exercise 20: $\lim_{x\to\infty}\dfrac{(x+1)^{\log(x+1)}}{x^{\log x}}$

Exercise 21: $\lim_{x\to\infty}\left(\dfrac{\log(x+1)}{\log x}\right)^x$

Exercise 22: $\lim_{x\to\infty}\dfrac{xe^{\sin x}}{\log x}$

5.3.5 An Important Limit: $\lim_{x\to\infty} x\left(1-\dfrac{x^p}{(x+1)^p}\right)$

Overview of Chapter 6: Integrals

Document: 6.1 Introducing Integrals as Antiderivatives

Movie: 6.1 Introducing Integrals as Antiderivatives

6.1.1 Preliminary Note: This Movie Takes the Fast Track into Integral Calculus

6.1.2 Defining Integrals Using Antiderivatives

Reviewing a Property of Antiderivatives
Defining the Symbol $\int_a^b f(x)dx$
Notation for Taking a Function Between Limits
The Symbol x is Not Important

6.1.3 Some Examples to Illustrate the Definition of an Integral

Example 1: $\int_2^5 xdx$

Example 2: $\int_0^{\pi/2} \cos xdx$

Example 3: $\int_2^9 \frac{1}{x}dx$

Exàmple 4: $\int_0^{\pi/4} \sec^2 xdx$

Example 5: $\int_0^{\pi/4} \sec x\tan xdx$

Example 6: $\int_0^{\pi/4} \sec xdx$

Example 7: $\int_{-1}^{2} 2x\sqrt{1+x^2}\,dx$

6.1.4 Linearity and Additivity of the Integral

Linearity of the Integral
Additivity of the Integral
The Symbol $\int_b^a$ when $a < b$

6.1.5 Using Integrals to Find Area

The Area Under the Graph of a Nonnegative Function: Historical Approach Using Infinitesimals
The Area Under the Graph of a Nonnegative Function Without Using Infinitesimals

Area of the Region Between Two Graphs
Area Between the Graph of a Negative Function and the x-Axis

6.1.6 Some Exercises on Area

Exercise 1: The region between $y = 4 - x^2$ and the x-axis
Exercise 2: A triangular region
Exercise 3: Region between $y = x^3 - 3x^2 + 2$ and $y = -x^2 + 3x + 2$
Exercise 4: Region between $y = \sin x$ and $y = \cos x$
Exercise 5: Region between $y = \sin x$ and $y = \sin 2x$
Exercise 6: Region between $y = \sin x$ and $y = \sqrt{\sin x}\,\cos x$

6.1.7 Derivatives of Integrals: The Equation $\frac{d}{dx}\int_a^x f(t)dt = f(x)$

6.1.8 Exercises on Derivatives of Integrals

Exercise 1: $\frac{d}{dx}\int_1^x \sqrt{1+t+t^4}\,dt$
Exercise 2: $\frac{d}{dx}\int_1^x \sqrt{1+t+t^4}\,dt$
Exercise 3: $\frac{d}{dx}\int_2^{\sin x} \sqrt{1+t+t^4}\,dt$
Exercise 4: $\frac{d}{dx}\int_2^{\log x} \sqrt[3]{1+\sin^2 t}\,dt$
Exercise 5: $\frac{d}{dx}\int_{\exp(\sin x)}^{5} \sqrt[3]{1+t^2}\,dt$
Exercise 6: $\frac{d}{dx}\int_{\sin x}^{\exp\left(x^2\right)} \sqrt{1+t^4}\,dt$

Document: 6.2 Riemann Sums

Movie: 6.2 Riemann Sums

6.2.1 Summation Notation

Introducing Summation Notation
Some Simple Examples to Illustrate Summation Notation

Example 1: $\sum_{j=3}^{5} j^3$

Example 2: $\sum_{j=0}^{7} (-1)^j$

Example 3: $\sum_{j=1}^{n} 4$

Arithmetical Rules for Summation
Working Out the Sum $\sum_{j=1}^{n} j$
Another Way of Working Out $\sum_{j=1}^{n} j$
Working Out the Sum $\sum_{j=1}^{n} j^2$
Working Out the Sum $\sum_{j=1}^{n} j^3$
Using a Computer Algebra System to Work Out $\sum_{j=1}^{n} j^p$

6.2.2 Introduction to Riemann Sums

Motivating Riemann Sums
Definition of a Partition
Definition of a Riemann Sum
Regular Partitions
Darboux's Theorem
Left Sums, Right Sums, and Midpoint Sums
- Left Sums
- Right Sums
- Midpoint Sums

6.2.3 Some Examples to Illustrate Darboux's Theorem

Example 1: $\int_0^1 x dx$

Example 2: $\int_0^1 x^2 dx$

Example 3: $\int_a^b x^2 dx$

Example 4: $\int_0^1 \sqrt{x}\, dx$

Example 5: $\int_0^1 \sqrt[3]{x^2}\, dx$

Document: 6.3 Riemann Sums with a Computer Algebra System

Movie: 6.3 Riemann Sums with a Computer Algebra System

6.3.1 Introductory Comment

6.3.2 Setting up The Riemann Sums

Supplying the Regular Partition to a Computer Algebra System
Introducing a Temporary Function f
Defining the Left Sum of a Function
Defining the Right Sum of a Function
Defining the Trapezoidal Sum of a Function
Defining Midpoint Sum of a Function
Defining the Simpson Sum of a Function
Motivation of the Simpson Sum

6.3.3 Numerical Approximations to Integrals

Summary of the Definitions
Using the Sums to Estimate $\int_0^1 \sqrt[3]{1+x^2}\, dx$
Using the Sums to Estimate $\int_0^1 \sqrt[3]{1-x^2}\, dx$
Obtaining Arrays of Approximating Sums Automatically

Document: 6.4 Using Riemann Sums to Define an Integral

Movie: 6.4 Using Riemann Sums to Define an Integral

6.4.1 Our Objective in this Section

6.4.2 A Quick Review of Riemann Sums

Bounded Functions
Definition of a Partition

Definition of a Riemann Sum
Regular Partitions

6.4.3 Squeezing a Function, Integrability, and the Integral

Motivating the Idea of Squeezing
Definition of a Squeezing Pair of Sequences
A Key Fact About a Squeezing Pair of Sequences
Integrability and the Integral

6.4.4 Some Examples to Illustrate Integrability

Example 1: The Integral $\int_0^1 x\,dx$
Example 2: The Integral $\int_0^1 x^2\,dx$
Example 3: Increasing Functions Are Integrable
Example 4: Decreasing Functions Are Integrable
Example 5: Continuous Functions Are Integrable
Example 6: A Function that Fails to be Integrable

6.4.5 Some Facts About the Integral

Linearity of the Integral
Nonnegativity of the integral
Additivity of the Integral
Darboux's Theorem

6.4.6 The Fundamental Theorem of Calculus

Part 1 of the Fundamental Theorem of Calculus
Part 2 of the Fundamental Theorem of Calculus

6.4.7 Optional Item: Error Estimates for the Simpson Sum (Not included in the video)

Background for the Error Estimates
An Example to Illustrate the Error Estimates

Overview of Chapter 7: Evaluating Integrals

Document: 7.1 Evaluating Integrals by Substitution

Movie: 7.1 Evaluating Integrals by Substitution

7.1.1 Some Common Antiderivatives

The Antiderivative $\int x^p\,dx$ when $p \neq -1$
The Antiderivative $\int e^x\,dx$
The Antiderivative $\int x^p\,dx$ when $p = -1$
The Antiderivative $\int \cos x\,dx$
The Antiderivative $\int \sin x\,dx$
The Antiderivative $\int \sec^2 x\,dx$
The Antiderivative $\int \sec x \tan x\,dx$
The Antiderivative $\int \tan x\,dx$
The Antiderivative $\int \cot x\,dx$
The Antiderivative $\int \sec x\,dx$
The Antiderivative $\int \frac{1}{\sqrt{1-x^2}}\,dx$
The Antiderivative $\int \frac{1}{1+x^2}\,dx$

The Antiderivative $\int \frac{1}{x\sqrt{x^2-1}}dx$

The List of Antiderivatives

7.1.2 Changing Variable to Calculate an Integral

Introducing the Change of Variable Method
Applying the Change of Variable Method

7.1.3 Some Exercises on the Change of Variable Method

Exercise 1: $\int_0^{\pi/2} \sin^2 x \cos x dx$

Exercise 2: $\int_0^1 \sqrt{1+x^2}\, 2x dx$

Exercise 3: $\int_0^1 \frac{4x+3}{\sqrt{2x^2+3x+7}} dx$

Exercise 4: $\int_1^2 \frac{x}{1+x^2} dx$

Exercise 5: $\int_0^{\pi} \cos^4 x \sin x dx$

Exercise 6: $\int_0^{\pi/4} \sqrt{\tan x}\, \sec^2 x dx$

Exercise 7: $\int_0^{\pi/3} \tan^2 x dx$

Exercise 8: $\int_0^{\pi/3} \tan^3 x dx$

Exercise 9: $\int_0^{\pi/4} \tan^4 x dx$

Exercise 10: $\int_1^{\exp(\pi/3)} \frac{\cos(\log x)}{x} dx$

Exercise 11: $\int_{\log(\pi/12)}^{\log(\pi/6)} e^x \sin(3e^x) dx$

Exercise 12: $\int_0^1 x\sqrt{x+3}\, dx$

Exercise 13: $\int_0^{\pi/2} \sqrt{\sin x}\, \cos x dx$

Exercise 14: $\int_0^{\pi/2} \sqrt{\sin x}\, \cos^3 x dx$

Exercise 15: $\int_0^{\pi/2} \sqrt{\sin x}\, \cos^5 x dx$

Exercise 16: $\int_0^{\pi/2} \sqrt{\cos x}\, \sin^5 x dx$

Exercise 17: $\int_0^{\pi/2} \cos^2 x dx$

Exercise 18: $\int_0^{\pi/2} \sin^4 x \cos^2 x dx$

Exercise 19: $\int_0^{\pi} \sqrt[3]{1+2\sin^2 x+\sin^5 x}\, \cos x dx$

Exercise 20: $\int_0^{\pi/4} \sec^6 x \sqrt{\tan x}\, dx$

Exercise 21: $\int_0^{\pi/3} \sec^3 x \tan^5 x dx$

Exercise 22: $\int_0^{\pi/3} \sec x dx$

Exercise 23: $\int_0^{1/2} \frac{\sqrt[3]{\arcsin x}}{\sqrt{1-x^2}} dx$

Exercise 24: $\int_1^{\sqrt{3}} \frac{1}{(1+x^2)\arctan x}dx$

Exercise 25: $\int_0^1 \frac{\arctan x}{(1+x^2)\sqrt{1+(\arctan x)^2}}dx$

Exercise 26: $\int_{\sqrt{2}}^2 \frac{1}{x\sqrt{x^2-1}\operatorname{arcsec} x}dx$

Document: 7.2 Evaluating Integrals by Parts

Movie: 7.2 Evaluating Integrals by Parts

7.2.1 Introduction to Integration by Parts

7.2.2 Some Examples to Illustrate Integration by Parts

Example 1: $\int_0^{\pi/2} x\cos x dx$

Example 2: $\int_0^1 xe^{3x}dx$

Example 3: $\int_0^{\pi/2} \cos^2 x dx$

7.2.3 Explaining Integration by Parts

Explaining Integration by Parts for Integrals
Explaining Integration by Parts for Antiderivatives

7.2.4 Exercises on Integration by Parts

Exercise 1

Exercise 1 Part a: $\int_0^{\pi/2} x^2\cos x dx$

Exercise 1 Part b: $\int x^2\cos x dx$

Exercise 2

Exercise 2 Part a: $\int_1^2 x\log x dx$

Exercise 2 Part b: $\int x\log x dx$

Exercise 3

Exercise 3 Part a: $\int_1^2 x(\log x)^2 dx$

Exercise 3 Part b: $\int x(\log x)^2 dx$

Exercise 4

Exercise 4 Part a: $\int_1^2 x(\log x)^3 dx$

Exercise 4 Part b: $\int x(\log x)^3 dx$

Exercise 5: $\int_1^2 \log x dx$

Exercise 6: $\int_0^{\pi^2/4} \cos\sqrt{x}\,dx$

Exercise 7: $\int_0^1 \arctan x dx$

Exercise 8: $\int_0^1 x\arctan x dx$

Exercise 9: $\int_0^{1/2} \arcsin x dx$

Exercise 10: $\int_0^{\pi/2} x \sin x \cos x dx$

Exercise 11: $\int_0^1 x \arcsin x dx$

Exercise 12: $\int_0^{\pi} e^x \cos x dx$

Exercise 13: $\int_0^{\pi/3} \sec^3 x dx$

Exercise 14: $\int_0^{\log\sqrt{3}} \operatorname{sech}^3 x dx$

Exercise 15: $\int_0^{2\pi} \cos mx \cos nx dx$

7.2.5 Reduction Formulas

Introduction to Reduction Formulas
Example 1: A Reduction Formula for the Integral $\int_1^2 x(\log x)^n dx$
Example 2: A Reduction Formula for the Antiderivative $\int x(\log x)^n dx$
Example 3: A Reduction Formula for the Integral $\int_0^{\pi/2} x^n \cos x dx$
Example 4: A Reduction Formula for the Antiderivative $\int x^n e^x dx$
Example 5: A Reduction Formula for the Antiderivative $\int \cos^n x dx$
Example 6: A Reduction Formula for the Integral $\int_0^{\pi/2} \cos^n x dx$
Example 7: A Reduction Formula for the Antiderivative $\int \sin^n x dx$
Example 8: A Reduction Formula for the Integral $\int_0^{\pi/2} \sin^n x dx$
Example 9: A Reduction Formula for the Antiderivative $\int \tan^n x dx$
Example 10: A Reduction Formula for the Antiderivative $\int \cot^n x dx$
Example 11: A Reduction Formula for the Antiderivative $\int \sec^n x dx$
Example 12: A Reduction Formula for the Integral $\int_0^{\pi/4} \sec^n x dx$

7.2.6 Wallis' Formula: $\lim_{n\to\infty} \frac{2^{2n}(n!)^2}{\sqrt{n}\,(2n)!} = \sqrt{\pi}$

Introduction to Wallis' Formula
A Return to the Integral $\int_0^{\pi/2} \cos^n x dx$
Deriving Wallis' Formula

Document:

7.3 Evaluating Integrals Using Trigonometric and Hyperbolic Substitutions

Movie Option 1:

7.3 Evaluating Integrals Using Trigonometric Substitutions Only

Movie Option 2:

7.3 Evaluating Integrals Using Trigonometric and Hyperbolic Substitutions

7.3.1 Preliminary Notes

Introduction to this Section
How Do I Know Whether to Use Trig or Hyperbolic Substitutions?
How Do I Know Whether a Given Integral Is of Type 1, 2, or 3?

7.3.2 Substitutions Involving $\sin$ *or* $\tanh$

Introduction to the $\sin$ Substitution
An Example to Illustrate the $\sin$ Substitution

Evaluation Using a Hyperbolic Substitution

Exercise 12: $\int_{3\sqrt{2}}^{6} \frac{1}{x\sqrt{x^2-9}}\,dx$

Evaluation Using a Trigonometric Substitution
Evaluation Using a Hyperbolic Substitution

Exercise 13: $\int_{3\sqrt{2}}^{6} \frac{1}{x^2\sqrt{x^2-9}}\,dx$

Evaluation Using a Trigonometric Substitution
Evaluation Using a Hyperbolic Substitution

Exercise 14: $\int_{3\sqrt{2}}^{6} \frac{1}{x^4\sqrt{x^2-9}}\,dx$

Evaluation Using a Trigonometric Substitution
Evaluation Using a Hyperbolic Substitution

Exercise 15: $\int_{3\sqrt{2}}^{6} \frac{1}{\sqrt{x^2-9}}\,dx$

Evaluation Using a Trigonometric Substitution
Evaluation Using a Hyperbolic Substitution

Exercise 16: $\int_{3\sqrt{2}}^{6} \frac{x}{\sqrt{x^2-9}}\,dx$

Exercise 17: $\int_{3\sqrt{2}}^{6} \frac{x^3}{\sqrt{x^2-9}}\,dx$

Exercise 18: $\int_{0}^{\pi/2} \frac{\cos x}{\sqrt{1+\sin^2 x}}\,dx$

Evaluation Using a Trigonometric Substitution
Evaluation Using a Hyperbolic Substitution

Exercise 19: $\int_{1}^{2} \frac{\sqrt{x^2-1}}{x^4}\,dx$

Evaluation Using a Trigonometric Substitution
Evaluation Using a Hyperbolic Substitution

Exercise 20: $\int_{5}^{3+2\sqrt{3}} \frac{1}{\sqrt{x^2-6x+13}}\,dx$

Evaluation Using a Trigonometric Substitution
Evaluation Using a Hyperbolic Substitution

Exercise 21: $\int_{1/2}^{2} \frac{1}{(2x^2-2x+5)^{3/2}}\,dx$

Evaluation Using a Trigonometric Substitution
Evaluation Using a Hyperbolic Substitution

Exercise 22: $\int_{1+3\sqrt{2}}^{7} \frac{1}{\sqrt{x^2-2x-8}}\,dx$

Evaluation Using a Trigonometric Substitution
Evaluation Using a Hyperbolic Substitution

Exercise 23: $\int_{3}^{5} \sqrt{6x-5-x^2}\,dx$

Evaluation Using a Trigonometric Substitution
Evaluation Using a Hyperbolic Substitution: **Omitted**

Exercise 24: $\int_{1}^{4/3} \frac{1}{(18x-9x^2-5)^{3/2}}\,dx$

Evaluation Using a Trigonometric Substitution
Evaluation Using a Hyperbolic Substitution

Document: 7.4 Integration of Rational Functions

Movie: ***7.4 Integration of Rational Functions***

7.4.1 Background on Rational Functions

Introducing Rational Functions

Partial Fraction Expansions of Rational Functions

7.4.2 Some Exercises on Integration of Rational Functions

Exercise 1: $\int \frac{x+23}{x^2-3x-10}\,dx$

Exercise 2: $\int_0^1 \frac{3x^2+8x+7}{(x+1)(x+2)^2}\,dx$

Exercise 3: $\int_{-1}^1 \frac{x^2+x+2}{(x+3)(x^2+2x+5)}\,dx$

Exercise 4: $\int_{-1}^1 \frac{2x-2}{(x+3)(x^2+2x+5)}\,dx$

Exercise 5: $\int_{-1}^1 \frac{x^2+5x-2}{(x+3)(x^2+2x+5)}\,dx$

Exercise 6: $\int_0^{\pi/4} \sqrt{\tan x}\,dx$

Exercise 7: $\int_0^{\pi/4} \sqrt[3]{\tan x}\,dx$

7.4.3 Integrating Rational Functions of $\cos$ *and* $\sin$

7.4.4 Exercises on Rational Functions of $\cos\theta$ *and* $\sin\theta$

Exercise 1: $\int_0^{\pi/2} \frac{1}{\sin\theta+\cos\theta}\,d\theta$

Exercise 2: An Alternative Approach to $\int_0^{\pi/2} \frac{1}{\cos\theta+\sin\theta}\,d\theta$

Exercise 3: $\int_0^{\pi/2} \frac{\sin\theta}{\sin\theta+\cos\theta}\,d\theta$

Exercise 4: $\int_0^{\pi/2} \frac{\sin\theta}{1+\cos\theta+\sin\theta}\,d\theta$

Document: 7.5 Evaluating Improper Integrals

Movie: ***7.5 Evaluating Improper Integrals***

7.5.1 Introduction to Improper Integrals

Example 1: Motivating The integral $\int_0^1 \frac{1}{\sqrt{x}}\,dx$

Example 2: Motivating The integral $\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$

Example 3: Motivating The integral $\int_1^\infty \frac{1}{x^2}\,dx$

Definition of an Improper Integral

Definition of an Integral that Improper at its Right Endpoint

Definition of an Integral that Improper at its Left Endpoint

Convergence and Divergence of an Improper Integral

7.5.2 Some Examples of Improper Integrals

Example 1: $\int_0^1 \frac{1}{\sqrt{x}}\,dx$

Example 2: $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$

Example 3: $\int_1^\infty \frac{1}{x^2}\,dx$

Example 4: $\int_0^{\infty} \cos x dx$

Example 5: $\int_0^{1} \arcsin x dx$

7.5.3 Some Exercises on Improper Integrals

Exercise 1: $\int_0^{\infty} \frac{1}{(1+x^2)^{3/2}} dx$

The More Careful Approach
The Quick Approach

Exercise 2: $\int_2^{\infty} \frac{1}{x\sqrt{x^2-1}} dx$

The More Careful Approach
The Quick Approach

Exercise 3: $\int_1^{2} \frac{1}{x\sqrt{x^2-1}} dx$

The More Careful Approach
The Quick Approach

Exercise 4: $\int_0^{\pi/2} \tan x dx$

The More Careful Approach
The Quick Approach

Exercise 5: $\int_0^{\pi/2} \sqrt{\tan x \sin x}\, dx$

The More Careful Approach
The Quick Approach

Exercise 6: $\int_0^{\pi/2} \frac{x\cos x - \sin x}{x^2} dx$

The More Careful Approach
The Quick Approach: **Omitted**

Exercise 7: $\int_0^{\infty} e^{-x} \sin x dx$

The More Careful Approach
The Quick Approach

Exercise 8: $\int_0^{1} \frac{1}{x^p} dx$

Exercise 9: $\int_1^{\infty} \frac{1}{x^p} dx$

Exercise 10: $\int_2^{\infty} \frac{1}{x(\log x)^p} dx$

Exercise 11: $\int_0^{2} \frac{1}{(x-1)^{1/3}} dx$

Exercise 12: $\int_0^{1} \log x dx$

Document: 7.6 Convergence of Improper Integrals

Movie: 7.6 Convergence of Improper Integrals

7.6.1 Introduction to This Section

7.6.2 Convergence of Integrals of Nonnegative Functions

An Fundamental Principle About Integrals of Nonnegative Functions
Warning
An Example to Illustrate The Fundamental Principle
Introducing The Comparison Test for Improper Integrals
The Comparison Test for Improper Integrals

An Example to Illustrate the Comparison Test
A Second Example to Illustrate the Comparison Test
Introduction to the Limit Version of the Comparison Test
Statement of the Limit Comparison Test
Another Way of Looking at the Limit Comparison Test

7.6.3 Exercises on the Comparison Test

Exercise 1: $\int_1^\infty \frac{x}{x^3 - 3x^2 + 3x + 7} dx$

Exercise 2: $\int_0^1 \frac{1}{\sqrt{x \cos x}} dx$

Exercise 3: $\int_1^\infty \frac{\log x}{x^2} dx$

Exercise 4: $\int_1^\infty \frac{1}{\sqrt[3]{x^2 + 5x + 2}} dx$

Exercise 5: $\int_0^1 \frac{\sin^2 x}{x^{5/2}} dx$

Exercise 6: $\int_1^\infty \frac{\sqrt{x}}{x^2 - x + 1} dx$

Exercise 7: $\int_0^{\pi/2} \sqrt{\tan x}\, dx$

Exercise 8: $\int_1^2 \frac{1}{\log x} dx$

Exercise 9: $\int_0^{\pi/2} \log(\sin x) dx$

Exercise 10: $\int_1^\infty x^{\alpha-1} e^{-x} dx$

Exercise 11: $\int_0^1 x^{\alpha-1} e^{-x} dx$

Note On the Final Three Exercises of this Group

Exercise 12: $\int_2^\infty \frac{1}{(\log x)^{\log x}} dx$

Exercise 13: $\int_3^\infty \frac{1}{(\log \log x)^{\log x}} dx$

Exercise 14: $\int_3^\infty \frac{1}{(\log x)^{\log \log x}} dx$

7.6.4 Improper Integrals of Functions that Can Change Sign

Absolute Convergence of an Improper Integral
Every Absolutely Convergent Integral Must Converge
Conditional Convergence of an Improper Integral

7.6.5 Exercises on Absolute and Conditional Convergence of Improper Integrals

Exercise 1: $\int_1^\infty \frac{\sin x}{x^2} dx$ and $\int_1^\infty \frac{\cos x}{x^2} dx$

Exercise 2: $\int_1^\infty \frac{\sin x}{x} dx$

Exercise 3: $\int_1^\infty \frac{\sin cx}{x^p} dx$

Exercise 4: $\int_1^\infty \frac{\cos cx}{x^p} dx$

Exercise 5: $\int_1^\infty \frac{\sin^2 x}{x} dx$

Exercise 6: $\int_1^\infty \frac{|\sin x|}{x} dx$

Exercise 7: Conditional Convergence of $\int_1^\infty \frac{\sin x}{x} dx$

Overview of Chapter 8: Some Applications of Derivatives and Integrals

Document: 8.1 Using Integrals to Find Volume

Movie: 8.1 Using Integrals to Find Volume

8.1.1 Volume by the Method of Slicing

8.1.2 Exercises on the Method of Slicing

Exercise 1: Volume of a Cone
Exercise 2: Volume of a Pyramid
Exercise 3: Volume of a Ball
Exercise 4: A Variation on the Cone Problem
Exercise 5: Rotating a Plane Region Around the x-Axis
Exercise 6: A Specific Region Rotated Around the x-Axis
Exercise 7: A Return to the Volume of a Ball Exercise
Exercise 8: Volume of a Bagel
Exercise 9: An Apple Without its Core
Exercise 10: Rotating a region bounded by $y = \sin x$ and $y = \cos x$ about the x-axis

8.1.3 Volume by the Method of Shells

Introducing the Shell Method
Finding the Volume of a Cylindrical Shell
Returning to Our Introduction

8.1.4 Exercises on the Method of Shells

Exercise 1
- Using the Slicing Method to Find this Volume
- Using the Shell Method to Find this Volume

Exercise 2
- Using the Slicing Method to Find this Volume
- Using the Shell Method to Find this Volume

Exercise 3

Exercise 4
- Using the Slicing Method to Find this Volume
- Using the Shell Method to Find this Volume

Exercise 5
- Using the Slicing Method to Find this Volume
- Using the Shell Method to Find this Volume

Exercise 6: Using the Shell Method to Find the Volume of a Bagel

Document: 8.2 Work Done by a Force

Movie: 8.2 Work Done by a Force

8.2.1 Work Done by a Constant Force

Introducing the Units of Work
Lifting a Mass Near the Surface of the Earth

8.2.2 Work Done by a Variable Force

Introducing the Formula for Work Done by a Variable Force

8.2.3 Exercises on Work Done by a Force

Movie: 8.5 Conic Curves

Overview of Chapter 9: Sequences and Series

Document: 9.1 Limits of Sequences

Movie: 9.1 Limits of Sequences

Exercise 9: The Limit $\lim_{n\to\infty} \frac{2^n}{n!}$

Exercise 10: The Limit $\lim_{n\to\infty} \frac{5^n}{n!}$

Exercise 11: The Limit $\lim_{n\to\infty} \frac{\log(n!)}{n^2}$

Exercise 12: The Important Limit $\lim_{n\to\infty} n\left(1 - \frac{n^p}{(n+1)^p}\right)$

9.1.4 Monotone Sequences

Introduction to Monotone Sequences

A Condition for an Increasing Sequence to Converge

A Final Note

Document: 9.2 An Intuitive Motivation of Infinite Series

Movie: 9.2 An Intuitive Motivation of Infinite Series

9.2.1 Our Objective in this Section

9.2.2 Some Examples to Illustrate Infinite Series

Example 1: The Sum $0 + 0 + 0 + 0 + 0 + 0 + \cdots$

Example 2: The Sum $1 + 1 + 1 + 1 + 1 + 1 + \cdots$

Example 3: Taking $a_n = \begin{cases} 1 & \text{if} \quad 1 \le n \le 4 \\ 0 & \text{if} \quad n \ge 5 \end{cases}$

Example 4: The Infinitely Repeating Decimal $0.\overline{1}$

Example 5: The Infinitely Repeating Decimal $0.\overline{9}$

Example 6: The Infinitely Repeating Decimal $0.\overline{473}$

Example 7: The Sum $1 + x + x^2 + x^3 + \cdots$ When $-1 < x < 1$

Example 8: The Sum $1 - x + x^2 - x^3 + x^4 - \cdots$ When $-1 < x < 1$

Example 9: The Sum $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$

Example 10: The sum $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$

Example 11: The sum $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$

Example 12: The sum $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$

Example 13: The Equation $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

Example 14: The Equation $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

Example 15: The Equation $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

Example 16: Comparing the Series Expansions of exp, cos, and sin

Example 17: The Equation $x^2 = \frac{\pi^2}{3} - \frac{4\cos x}{1^2} + \frac{4\cos 2x}{2^2} - \frac{4\cos 3x}{3^2} + \cdots$

9.2.3 Concluding Remarks

Document: 9.3 Introduction to Infinite Series

Movie: 9.3 Introduction to Infinite Series

9.3.1 The Series with nth Term a_n

9.3.2 Convergence and Divergence of Series

9.3.3 Some Examples to Illustrate the Idea of a Series

Example 1: The Series $\sum 0$
Example 2: The Series $\sum 1$
Example 3: Taking $a_n = \begin{cases} 1 & \text{if} \quad 1 \le n \le 4 \\ 0 & \text{if} \quad n \ge 5 \end{cases}$
Example 4: The Series $\sum \frac{1}{n(n+1)}$
Example 5: The Series $\sum \frac{2}{n(n+1)(n+2)}$ for Each n
Example 6: The Geometric Series $\sum x^{n-1}$
Example 7: The Series $\sum \log\left(1+\frac{1}{n}\right)$
Example 8: The Series $\sum (-1)^{n-1}$

9.3.4 The nth Term Criterion for Divergence

Introduction to the nth Term Criterion for Divergence
Proof of the nth Term Criterion for Divergence

9.3.5 A Return to the Examples of 9.3.3

Example 1: The Series $\sum 0$
Example 2: The Series $\sum 1$
Example 3: Taking $a_n = \begin{cases} 1 & \text{if} \quad 1 \le n \le 4 \\ 0 & \text{if} \quad n \ge 5 \end{cases}$
Example 4: The Series $\sum \frac{1}{n(n+1)}$
Example 5: The Series $\sum \frac{2}{n(n+1)(n+2)}$
Example 6: The Geometric Series $\sum x^{n-1}$
Example 7: The Series $\sum \log\left(1+\frac{1}{n}\right)$
Example 8: The Series $\sum (-1)^{n-1}$

9.3.6 Some Applications of the nth Term Criterion for Divergence

A Ratio Criterion for Divergence
Testing the Series $\sum \frac{n!}{6^n}$
Divergence of the Series $\sum \frac{(2n)!}{(n!)^2}$
Divergence of The Series $\sum \frac{(-1)^n 4^n (n!)^2}{(2n)!}$
A Problem that We Cannot Solve Right Now: Test the Series $\sum \frac{(2n)!}{4^n (n!)^2}$
A Limit Form of the Ratio Criterion for Divergence
Divergence of the Series $\sum \frac{3^n}{n^{10}}$
Divergence of the Series $\sum \frac{(3^n)(n!)}{n^n}$

9.3.7 A Quick Summary of What We Know at Present

Document: 9.4 Convergence of Nonnegative Series

Movie: 9.4 Convergence of Nonnegative Series

9.4.1 Introduction to Nonnegative Series

9.4.2 The Integral Comparison Test

Divergence of the Series $\sum \frac{1}{n}$

Convergence of the Series $\sum \frac{1}{n^2}$

The General Form of the Integral Comparison Test

The p-Series

- The p-Series When $p > 1$
- The p-Series When $p < 1$
- Conclusion: Convergence Criteria for the p-Series

A Sharper Form of the p-Series

- The Case $p = 1$
- The Case $p < 1$
- The Case $p > 1$

9.4.3 Optional: A Sharper Type of Integral Comparison

An Extension of the Integral Comparison Test

Euler's Constant

The Limit $\lim_{n\to\infty} \sum_{j=n+1}^{2n} \frac{1}{j}$

Summing the Series $\sum \frac{(-1)^{n-1}}{n}$

Summing the Series $\sum \frac{1}{n(2n-1)}$

9.4.4 Comparing Series with One Another

The Comparison Test: Inequality Form

The Comparison Test: Limit Form

9.4.5 Some Exercises on The Comparison Test

Exercise 1: Testing the Series $\sum \frac{\sin^2 n}{n^2}$

Exercise 2: An Unsuccessful Attempt to Test the Series $\sum \frac{\sin^2 n}{n}$

Exercise 3: Testing the Series $\sum \frac{n}{n^4+7}$

Exercise 4: Testing the Series $\sum \frac{n}{n^4-7}$

Exercise 5: Testing the Series $\sum \frac{1}{n^{3/2}+n}$

Exercise 6: Testing the Series $\sum \frac{1}{n^{3/2}-n}$

Exercise 7: Testing the Series $\sum \frac{n}{\sqrt{n^4-n^2+2}}$

Exercise 8: Testing the Series $\sum \frac{\log n}{n^2}$

Exercise 9: Testing the Series $\sum \frac{n \log n}{\sqrt{n^5-n^2+2}}$

Exercise 10: Testing the Series $\sum \frac{1}{n^{1+1/n}}$

Exercise 11: Testing the Series $\sum \frac{1}{n^{1+(\log n)/n}}$

Exercise 12: Testing the Series $\sum \frac{1}{n^{1+(\log n)^2/n}}$

Exercise 13: Testing the Series $\sum \left(\frac{n}{n+1}\right)^n$

Exercise 14: Testing the Series $\sum \left(\frac{1}{\log n}\right)^3$

Exercise 15: Testing the Series $\sum \left(\frac{1}{\log n}\right)^n$

Exercise 16: Testing the Series $\sum \left(\frac{1}{\log n}\right)^{\log n}$

Exercise 17: Testing the Series $\sum \left(\frac{1}{\log \log n}\right)^{\log n}$

Exercise 18: Testing the Series $\sum \left(\frac{1}{\log n}\right)^{\log \log n}$

9.4.6 The Elementary Ratio Tests

Introducing the Ratio Tests
The Ratio Comparison Test
The d'Alembert Ratio Test, Inequality Form
The d'Alembert Ratio Test, Limit Form, Often Known as "The Ratio Test"

9.4.7 Some Exercises that Rely on d'Alembert's Test (Exercises on "The Ratio Test")

Exercise 1: Testing the Series $\sum \frac{n^{1000000}}{2^n}$

Exercise 2: Testing the Series $\sum \frac{2^n}{n!}$

Exercise 3: Testing the Series $\sum \frac{n!}{n^n}$

Exercise 4: Testing the Series $\sum \frac{n^{cn}}{n!}$ Given $c < 1$

Exercise 5: Testing the Series $\sum \frac{(2^n)(n!)}{n^n}$

Exercise 6: Testing the Series $\sum \frac{(3^n)(n!)}{n^n}$

Exercise 7: An Unsuccessful Attempt to Test the Series $\sum \frac{(e^n)(n!)}{n^n}$

Exercise 8: An Unsuccessful Attempt to Test the Series $\sum \frac{n^n}{(e^n)(n!)}$

Exercise 9: Testing the Series $\sum \frac{(2n)!}{5^n (n!)^2}$

Exercise 10: Testing the Series $\sum \frac{(2n)!}{3^n (n!)^2}$

Exercise 11: Testing the Series $\sum \frac{4^n (n!)^2}{(2n)!}$

Exercise 12: An Unsuccessful Attempt to Test the series $\sum \frac{(2n)!}{4^n (n!)^2}$

Exercise 13: Testing the Series $\sum \frac{((2n)!)^3}{((3n)!)^2}$

Exercise 14: Testing the Series $\sum \frac{(\log n)^n}{c^n (\log 2)(\log 3)\cdots(\log n)}$ for $c > 0$

Exercise 15: A Second Visit to the Series $\sum \frac{(e^n)(n!)}{n^n}$

Exercise 6: A Relationship Between $\sum a_n$ and $\sum a_n^3$

9.5.7 Some Further Series to Test with the Alternating Series Test or Dirichlet's Test (Optional)

Introducing This Topic

A Special Technique for Testing Alternating Series

Testing the Series $\sum \frac{(-1)^n (2n)!}{4^n (n!)^2}$

Testing the Series $\sum \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!}$

Testing the Series $\sum \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{n}\right)(-1)^n$

Testing the Series $\sum \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{n^2}\right)(-1)^n$

Document: 9.6 Power Series

Movie: **9.6 Power Series**

9.6.1 Introduction to Power Series

9.6.2 Some Examples of Power Series

Example 1: The Geometric Series $\sum x^n$

Example 2: The Series $\sum \frac{x^n}{n!}$

Example 3: The Series $\sum \frac{(-1)^{n-1} x^n}{n}$

Example 4: The Series $\sum \frac{(-1)^{n-1} (x-4)^{2n-1}}{2n-1}$

Example 5: The Series $\sum n(x+3)^{n-1}$

Example 6: The Series $\sum (n!) x^n$

9.6.3 Radius and Interval of Convergence of a Power Series

The Case $0 < r < \infty$

The Case $r = 0$

The Case $r = \infty$

9.6.4 Some Exercises on Radius and Interval of Convergence

Exercise 1: The Series $\sum \frac{(x-5)^n}{2^n n^2}$

Exercise 2: The Series $\sum \frac{(x-5)^n}{2^n n}$

Exercise 3: The Series $\sum \frac{(-1)^n (x-5)^n}{2^n n}$

Exercise 4: The Series $\sum \frac{n(x-5)^n}{2^n}$

Exercise 5: The Series $\sum \frac{(x-5)^{2n}}{n 3^n}$

Exercise 6: The Series $\sum \frac{(n!)^2}{(2n)!} x^n$

Exercise 7: The Series $\sum \frac{(2n)!}{(n!)^2} x^n$

Exercise 8: The Binomial Series $\sum \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!} x^n$

The chapters below are under construction.

They may lack polish and their video versions are not yet implemented.

Overview of Chapter 10: Some Basics in Linear Algebra

Document: 10.1 A Glance at Second and Third Order Determinants

Movie: ***10.1 A Quick Look at Second and Third Order Determinants***

10.1.1 Second Order Determinants

Definition of a Second Order Determinant
Example 1
Example 2
Solving Two Equations for Two Unknowns

10.1.2 Third Order Determinants

Definition of a Third Order Determinant
Alternative Expansions of a Third Order Determinant
Example of a Third Order Determinant
Solving Three Equations for Three Unknowns

10.1.3 More General Determinants

Document: 10.2 Vectors in Space

Movie: ***10.2 Vectors in Space***

10.2.1 Preliminary Note

10.2.2 Introducing the Arithmetical Operations in $\boldsymbol{R}^n$

Definition of the Space $\boldsymbol{R}^n$
Addition and Subtraction in $\boldsymbol{R}^n$

10.2.3 Some Properties of Addition and Subtraction in $\boldsymbol{R}^n$

Adding any Point to the Origin
The Commutative Law for Addition in $\boldsymbol{R}^n$
The Associative Law for Addition in $\boldsymbol{R}^n$
Some Facts About Subtraction
The Symbol $-A$

10.2.4 Scalar Multiplication in $\boldsymbol{R}^n$

Introducing Scalar Multiplication
Definition of Scalar Multiplication in $\boldsymbol{R}^n$
Some Properties of Scalar Multiplication in $\boldsymbol{R}^n$

10.2.5 Linear Combinations

Definition of a Linear Combination
Two Examples of Linear Combinations
Example 1

Exercise 1: Equation of a Plane Containing a Given Point and Perpendicular to a Given Direction
Exercise 2: Equation of a Plane Containing Three Given Points
Exercise 3: Equation of a Line Containing Two Given Points
Exercise 4: Equation of a Line Containing a Given Point and Having a Given Direction
Exercise 5: Intersection of a Line and a Plane
Exercise 6: Failure of Intersection of a Line and a Plane
Exercise 7: Intersection of Two Lines
Exercise 8: Angle Between Two Given Lines
Exercise 9: Plane Containing Two Given Lines
Exercise 10: Plane Containing Three Given Points
Exercise 11: Plane Containing a Line and a Point
Exercise 12: Line Perpendicular to Two Given Lines
Exercise 13: Dropping a Perpendicular to a Line
Exercise 14: Point in a Line Closest to a Given Point
Exercise 15: Common Perpendicular Between Two Lines
Exercise 16: Perpendicular from a Point to a Plane

10.3.5 Parametric Equation of a Plane in $\boldsymbol{R}^3$

Introducing the Parametric Equation of a Plane
An Example of a Parametric Equation of a Plane

Overview of Chapter 11: Multivariable Differential Calculus

Document: 11.1 Surfaces and Curves in $\boldsymbol{R}^3$

Movie: ***11.1 Surfaces and Curves in $\boldsymbol{R}^3$***

11.1.1 Preliminary Note on This Section

11.1.2 Surfaces as Implicit Plots and Parametric Surfaces

11.1.3 Some Examples of Surfaces

Example 1: Plotting a Cone
Example 2: Plotting a Circular Parabaloid
Example 3: Plotting an Ellipsoid
Example 4: Plotting a Cone and a Hemisphere
Example 5: Plotting an Hyperboloid of One Sheet
Example 6: Plotting an Hyperboloid of Two Sheets
Example 7: Plotting a Corkscrew
Example 8: Plotting A Double Sea Shell
Example 9: Plotting a Cylinder
Example 10: Plotting Möbius Band
Example 11: Plotting a Cylinder with Two Twists
Example 12: Plotting a Cylinder with Three Twists
Example 13: Plotting a Cylinder with Four Twists
Example 14: Twisting a Cylinder
Example 15: A Surface with a Surprise

11.1.4 Parametric Curves

Motivating the Idea of a Parametric Curve in $\boldsymbol{R}^3$
11.1.4.2 Definition of a Parametric Curve in $\boldsymbol{R}^3$

11.1.5 Some Examples of Curves

Example 1: Plotting a Spiral on a Cylinder
Example 2: Plotting a Spiral on a Cone
Example 3: Plotting an Exponential Spiral
Example 4: Plotting Two Interlocking Closed Curves
Example 5: Plotting the Hardy-Walker Knotted Closed Curve

Document: 11.2 The Calculus of Curves

Movie: 11.2 The Calculus of Curves

11.2.1 Limits and Continuity of Parametric Curves

Limit of a Parametric Curve at a Given Number
Continuity of a Curve

11.2.2 Some Examples to Illustrate Limits and Continuity of Curves

Example 1
Example 2
Example 3

11.2.3 Velocity, also called the Derivative of a Curve

Definition of the Velocity of a Curve
Speed of a Curve
Acceleration of a Curve
An Example to Illustrate the Velocity, Speed and Acceleration of a Curve

11.2.4 Geometric Interpretation of Velocity and Speed

The Direction of the Velocity of a Curve
Using Speed to Find the Length of a Curve

11.2.5 Some Exercises on Velocity and Speed

Exercise 1
Exercise 2
Exercise 3
Exercise 4
Exercise 5
Exercise 6
Exercise 7
Exercise 8

11.2.6 Curvature, Principal Normal, Binormal, and Torsion of a Curve

Velocity of a Curve Whose Norm is Constant
Unit Tangent Vector of a Parametric Curve
Principal Normal of a Parametric Curve
The Curvature of a Parametric Curve
The Equation $T'(t) = k(t)s'(t)N(t)$
The Curvature of a Circle is the Reciprocal of Its Radius
Center of Curvature and Evolute of a Parametric Curve
The Binormal of a Parametric Curve
The Orthonormal Triple $\{T(t), N(t), B(t)\}$
The Torsion of a Parametric Curve

Definition of a Vector Field
Scalar Fields

11.5.2 Some Examples of Vector Fields

Example 1
Example 2
Example 3
Example 4

11.5.3 Gradient, Divergence, Laplacian, and Curl

Gradient of a Real Function
Gradient of a Real Function
The Laplacian
The Curl of a Vector Field
The Operator ∇ Called Nabla or Del

11.5.4 Exercises on Gradient, Curl, and Divergence

Exercise 1
Exercise 2
Exercise 3
Exercise 4
Exercise 5
Exercise 6
Exercise 7
Exercise 8

11.5.5 Conservative Vector Fields and Potential of a Field

Potential of a Vector Field
A Necessary Condition a Vector Field to be Conservative

11.5.6 Exercises on Conservative Fields and Potential

Exercise 1
Exercise 2
Exercise 3
Exercise 4
Exercise 5

11.5.7 Directional Derivative

Motivating the Idea of a Directional Derivative
Definition of the Directional Derivative of a Scalar Field
A Useful Formula for a Directional Derivative

11.5.8 Exercises on Directional Derivatives

Exercise 1
Exercise 2
Exercise 3

Document: 11.6 Further Topics on Partial Differentiation

Movie: *11.6 Further Topics on Partial Differentiation*

11.6.1 A Quick Look at Matrix Arithmetic

Definition of Local Maximum and Local Minimum of a Function

11.7.2 Some Examples to Illustrate the Definitions

Example 1
Example 2
Example 3
Example 4

11.7.3 Basic Facts About Maxima and Minima

Existence of Maxima and Minima of a Function
Fermat's Theorem
Critical Points of a Function
Finding Maxima and Minima of a Given Function
Saddle Points
The Second Derivative Test for Maxima and Minima

11.7.4 Exercises on Maxima and Minima

Exercise 1
Exercise 2
Exercise 3
Exercise 4
Exercise 5: A Box Problem
Exercise 6: A Maximum Minimum Problem that Requires a Computer Algebra System

11.7.5 The Standard Simplex in $\boldsymbol{R}^n$

The Standard Simplex in $\boldsymbol{R}^1$, $\boldsymbol{R}^2$, and $\boldsymbol{R}^3$
Definition of the Standard Simplex Q^n
A Maximum Minimum Problem on the Simplex Q^n
An Application of the Preceding Maximum Minimum Problem

Overview of Chapter 12: Multivariable Integral Calculus

Document: 12.1 Integration on Curves

Movie: 12.1 Integration on Curves

12.1.1 Integration on a Smooth Curve

Definition of a Smooth Curve
Integrals of the Type $\int_P f dx$, $\int_P f dy$, and $\int_P f dz$
Integrals of the Type $\int_P F \cdot dP = \int_P F \cdot (dx, dy, dz) = \int_P f dx + g dy + h dz$
Application to Work Done by a Force

12.1.2 Examples of Integrals on Smooth Curves

Example 1
Example 2
Example 3

12.1.3 Fundamental Theorem of Calculus for Integrals on Curves

Introduction to the Fundamental Theorem
Statement of the Fundamental Theorem for Integrals of the Type $\int_P F \cdot dP$