

# Virtual Calculus Tutor

## Table of Contents: Level 4

[Index](#)                      [How to Watch the Movies](#)  
[Preface](#)                      [Licence Information](#)  
[Check for Upgrade](#)      [About Virtual Calculus Tutor](#)

 Document:

### Overview of Chapter 1: Introduction to the Real Number System

Movie:  1 Introduction to the Real Number System

 **1.1 Philosophical Introduction to the System  $R$**

1.1.1 *What is a Number?*

1.1.2 *Numbers as Seen in Modern Mathematics*

 **1.2 An Intuitive Introduction to the System  $R$**

1.2.1 *Rational Numbers: the Numbers We See in Childhood*

1.2.2 *The Pythagorean Crisis*

1.2.3 *The World of Surds*

1.2.4 *In Search of a Complete Real Number System*

 **Overview of Chapter 2: Limits and Continuity**

 **Document: 2.1 Motivating the Idea of Slope of a Curved Graph**

Movie:  2.1 Motivating the Idea of Slope of a Curved Graph

 **2.1.1 Quick Review of Slopes of Straight Lines**

Slope of a Line Segment

The Slope of the Line  $y = 3x - 7$

The Slope of the Line  $y = mx + b$

 **2.1.2 Searching for the Meaning of Slope of a Curved Graph**

Introducing the Problem

An Example of a Curved Graph

Approximating the Slope of the Graph

 **2.1.3 Exercises on Numerical Approximation of Slopes**

Exercise 1: Slope of  $y = 2^x$  at  $(0, 1)$

Exercise 2: Slope of  $y = \frac{2^x}{1+x^2}$  at  $(1, 1)$

Exercise 3: Slope of  $y = \sin x$  at  $(\frac{\pi}{3}, \frac{\sqrt{3}}{2})$

Exercise 4: Slope of  $y = |x^2 - 9|$  at  $(3, 0)$  is undefined

Exercise 5: Slope of  $y = x \sin \frac{1}{x}$  at  $(0, 0)$  is undefined

Exercise 6: Slope of  $y = x^2 \sin \frac{1}{x}$  at  $(0, 0)$

## Document: 2.2 Introduction to the Limit Concept

Movie:  2.2 Introduction to the Limit Concept

### 2.2.1 Motivating the Idea of a Limit

### 2.2.2 Intuitive Definition of a limit

Example 1:  $f(t) = \frac{3^t - 9}{t - 2}$  for  $t \neq 2$

Example 2:  $g(x) = \frac{3^x - 9}{x - 2}$  for  $x \neq 2$

Example 3:  $f(x) = \begin{cases} \frac{3^x - 9}{x - 2} & \text{if } x \neq 2 \\ 6 & \text{if } x = 2 \end{cases}$

Example 4:  $f(x) = \frac{x^2 - 9}{x - 3}$  for  $x \neq 3$

Example 5:  $f(x) = x + 3$  for all  $x$

Example 6:  $f(x) = \begin{cases} x + 3 & \text{if } x \neq 3 \\ 4 & \text{if } x = 3 \end{cases}$

Example 7:  $f(x) = \begin{cases} x - 1 & \text{if } x < 3 \\ 5 - x & \text{if } x > 3 \end{cases}$

Example 8:  $(x) = \begin{cases} x - 1 & \text{if } x < 3 \\ 2 - x & \text{if } x > 3 \end{cases}$

Example 9:  $f(x) = \begin{cases} 2 + 3x & \text{if } x < 0 \\ \sin \frac{1}{x} & \text{if } x > 0 \end{cases}$

### 2.2.3 Limit Notation

The Symbol  $\lim$

Limits from the Left and Limits from the Right

Return to Example 7

Return to Example 8

### 2.2.4 Some Exercises on Limits

Exercise 1: Numerical approach to  $\lim_{x \rightarrow 1} \frac{\log_3 x}{x - 1}$

Exercise 2: Numerical approach to  $\lim_{u \rightarrow 0} \frac{\cos 3u - \cos 5u}{u^2}$

Exercise 3: Numerical approach to  $\lim_{x \rightarrow 1} \frac{(x(2^x) - 2)|x - 1|}{(x - 1)^2}$

Exercise 4: Numerical search for  $a$  to make  $\lim_{x \rightarrow 0} \frac{a^x - 1}{x} = 1$

## Document: 2.3 Properties of Limits

### Movie: 2.3 Properties of Limits

#### 2.3.1 Some Basic Facts

Limit of a Constant Function

The Equation  $\lim_{t \rightarrow x} t = x$

#### 2.3.2 The Arithmetical Rules

Limit of a Sum

Limit of a Difference

Limit of a Product

Limit of a Quotient

Limit of an Exponential Expression

#### 1.3.3 Using the Arithmetical Rules to Evaluate Limits

Example 1: Limit of a One Term Polynomial (Monomial)

Example 2: Limit of a Polynomial

Example 3: Limit of a Rational Function

Example 4: Limits and Exponents

Some Harder Limits

#### 2.3.4 Exercises that Make Use of the Arithmetical Rules

Exercise 1:  $\lim_{t \rightarrow 2} \frac{t - 1}{t - 2}$

Exercise 2:  $\lim_{t \rightarrow 2} \frac{t^3 - 8}{t - 2}$

Exercise 3:  $\lim_{t \rightarrow x} \frac{t^5 - x^5}{t - x}$

Exercise 4:  $\lim_{t \rightarrow x} \frac{t^{11} - x^{11}}{t - x}$

Exercise 5:  $\lim_{t \rightarrow x} \frac{t^{11} - x^{11}}{t^7 - x^7}$

Exercise 6:  $\lim_{t \rightarrow x} \frac{\sqrt[3]{t} - \sqrt[3]{x}}{t - x}$

Exercise 7:  $\lim_{t \rightarrow x} \frac{t^{3/5} - x^{3/5}}{t - x}$

Exercise 8:  $\lim_{t \rightarrow x} \frac{t^{-3} - x^{-3}}{t - x}$

Exercise 9:  $\lim_{t \rightarrow x} \frac{t^{-4/7} - x^{-4/7}}{t - x}$

Exercise 10:  $\lim_{t \rightarrow x} \frac{\frac{t}{1+t^2} - \frac{x}{1+x^2}}{t - x}$

#### 2.3.5 The Sandwich Rule

Stating the Sandwich Rule

Example to Illustrate the Sandwich Rule

#### 2.3.6 Infinite Limits

Introducing the Idea  $\lim_{t \rightarrow x} f(t) = \infty$

Introducing the Idea  $\lim_{t \rightarrow x} f(t) = -\infty$

#### 2.3.7 Examples To Illustrate Infinite Limits

Example 1:  $\lim_{t \rightarrow 3} \frac{1}{(t - 3)^2}$

Example 2:  $\lim_{t \rightarrow 3} \frac{-1}{(t-3)^2}$

Example 3:  $\lim_{t \rightarrow 3} \frac{1}{|t-3|}$

Example 4:  $\lim_{t \rightarrow 3} \frac{-1}{|t-3|}$

Example 5:  $\lim_{t \rightarrow 3^+} \frac{1}{t-3}$

Example 6:  $\lim_{t \rightarrow 3^-} \frac{1}{t-3}$

Example 7:  $\lim_{t \rightarrow 3} \frac{1}{t-3}$

### 2.3.8 Limits at $\infty$ and $-\infty$

Introducing the idea  $\lim_{x \rightarrow \infty} f(x)$

Introducing the idea  $\lim_{x \rightarrow -\infty} f(x)$

### 2.3.9 Examples on Limits at $\infty$ and $-\infty$

Example 1:  $\lim_{x \rightarrow \infty} \frac{1}{x}$  and  $\lim_{x \rightarrow -\infty} \frac{1}{x}$

Example 2:  $\lim_{x \rightarrow \infty} \frac{x}{x+1}$

Example 3:  $\lim_{x \rightarrow \infty} \frac{x}{x^2+1}$

Example 4:  $\lim_{x \rightarrow \infty} \frac{3x^2+x-5}{4x^2-8x+1}$

Example 5:  $\lim_{x \rightarrow \infty} \frac{\sqrt[3]{5x^6+2x^3-4x^2+x+3}}{\sqrt{2x^4+3x^2+4}}$

Example 6:  $\lim_{x \rightarrow \infty} (\sqrt{2x+1} - \sqrt{2x-3})$

Example 7:  $\lim_{x \rightarrow \infty} (\sqrt{x^2+3x+2} - \sqrt{x^2-3x+2})$

Example 8:  $\lim_{x \rightarrow \infty} \frac{\sqrt{x^4+2x^3+3} - \sqrt{x^4-2x^3+3}}{x}$

## Document: 2.4 Trigonometric Limits

Movie:  2.4 Trigonometric Limits

### 2.4.1 Radian Measure and Area of a Circular Sector

The Number  $\pi$

Radian Measure of an Angle

Area of a Circular Sector

Evaluating Trigonometric Functions at a Number

### 2.4.2 A Fundamental Trigonometric Inequality

The Case  $\theta$  Positive

The Case  $\theta$  Negative

Combining the Two Cases

### 2.4.3 Obtaining the Trigonometric Limits

Intuitive Approach to  $\lim_{\theta \rightarrow 0} \cos \theta$

Optional More Careful Approach to  $\lim_{\theta \rightarrow 0} \cos \theta$

The Limit  $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta}$

The Limit  $\lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta}$

### 2.4.4 Exercises on the Trigonometric Limits

- Exercise 1:  $\lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta^2}$
- Exercise 2:  $\lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta \sin \theta}$
- Exercise 3:  $\lim_{\theta \rightarrow 0} \frac{\sin 3\theta}{\theta}$
- Exercise 4:  $\lim_{\theta \rightarrow 0} \frac{\sin 5\theta}{\sin 4\theta}$
- Exercise 5:  $\lim_{\theta \rightarrow 0} \frac{\tan 3\theta}{\theta}$
- Exercise 6:  $\lim_{\theta \rightarrow 0} \frac{\sin 5\theta - \sin 3\theta}{\theta}$
- Exercise 7:  $\lim_{\theta \rightarrow 0} \frac{\cos 4\theta - \cos 6\theta}{\theta^2}$
- Exercise 8:  $\lim_{\theta \rightarrow 0} \frac{\sec \theta - \cos \theta}{\theta^2}$
- Exercise 9:  $\lim_{\theta \rightarrow 0} \frac{\tan \theta - \sin \theta}{\theta^3}$
- Exercise 10:  $\lim_{\theta \rightarrow 0} \frac{1 - \sqrt[3]{\cos \theta}}{\theta^2}$
- Exercise 11:  $\lim_{\theta \rightarrow 0} \frac{\sqrt[3]{\cos 3\theta} - \sqrt[3]{\cos 5\theta}}{\theta^2}$
- Exercise 12:  $\lim_{x \rightarrow 0^+} \sin \frac{1}{x}$
- Exercise 13:  $\lim_{x \rightarrow 0} x \sin \frac{1}{x}$

## Document: 2.5 Continuity

### Movie: 2.5 Continuity

#### 2.5.1 Introducing the Concept of Continuity

Review of the Intuitive Definition of a Limit  
 Definition of Continuity of a Function  $f$  at a Number  $x$

#### 2.5.2 Some Examples to Illustrate the Idea of a of Continous Function

Example 1:  $f(t) = 3t^2 - t + 2$  for all  $t$

Example 2:  $f(t) = \frac{t^2 + 4t - 2}{t^3 + 3t^2 - t + 4}$  when  $t^3 + 3t^2 - t + 4 \neq 0$

Example 3:  $f(t) = t + 3$  for  $t \neq 3$

Example 4:  $f(t) = \frac{t^2 - 9}{t - 3}$  when  $t - 3 \neq 0$

Example 5:  $f(t) = \begin{cases} \frac{t^2 - 9}{t - 3} & \text{if } t \neq 3 \\ 6 & \text{if } t = 3 \end{cases}$

Example 6:  $f(t) = \begin{cases} \frac{t^2 - 9}{t - 3} & \text{if } t \neq 3 \\ 2 & \text{if } t = 3 \end{cases}$

Example 7:  $f(t) = \begin{cases} \frac{t^2 - 9}{t - 3} & \text{if } t < 3 \\ 6 & \text{if } t = 3 \end{cases}$

Example 8:  $f(t) = \begin{cases} t + 3 & \text{if } t < 3 \\ 6 & \text{if } t = 3 \\ 2 - t & \text{if } t > 3 \end{cases}$

### 2.5.3 Properties of Continuous Functions

Preliminary Comment  
The Bolzano Intermediate Value Theorem  
    Introduction to the Bolzano Intermediate Value Theorem  
    Statement of the Bolzano Intermediate Value Theorem  
    More General Version of the Bolzano Intermediate Value Theorem  
    The Intermediate Value Property  
Maxima and Minima of Continuous Functions  
The Theorem on Existence of Maxima and Minima of Continuous Functions

### 2.5.4 Some Examples of Functions that Fail to Have a Maximum or a Minimum

The Effect of a Missing Endpoint  
The Effect of a Discontinuity

### 2.5.5 Exercises on the Properties of Continuous Functions

Exercise 1:  $f(x) = x^2$  for  $-3 \leq x \leq 3$

Exercise 2:  $f(x) = x^2$  for  $-3 < x < 3$

Exercise 3:  $f(x) = \begin{cases} x & \text{if } 0 < x < 2 \\ x - 2 & \text{if } 2 \leq x \leq 4 \end{cases}$

Exercise 4:  $f(x) = |x^2 - 4|$  for  $0 \leq x \leq 5/2$

Exercise 5:  $f(x) = \begin{cases} x & \text{if } 0 \leq x < 1 \\ 1 + 4x - x^2 & \text{if } 1 \leq x \leq 4 \end{cases}$

Exercise 6: Existence of a solution of  $5\sqrt[3]{x} + \sqrt{9-x} = 6$

## Overview of Chapter 3: Derivatives

### Document: 3.1 Introduction to Derivatives

Movie:  3.1 Introduction to Derivatives

#### 3.1.1 Definition of a Derivative

Motivating the Definition Using Slopes  
Definition of the Derivative of a Function  
Alternative Form of the Definition of a Derivative

#### 3.1.2 Some Examples of Derivatives

Example 1: Derivative of a constant  
Example 2:  $f(x) = mx + b$  for all  $x$   
Example 3:  $f(x) = x^2$  for all  $x$ , find  $f'(3)$   
Example 4:  $f(x) = x^2$  for all  $x$ , find  $f'(x)$   
Example 5:  $f(x) = x^3$  for all  $x$ , find  $f'(x)$   
Example 6:  $f(x) = x^7$  for all  $x$ , find  $f'(x)$

#### 3.1.3 The Power Rule

Introducing the Power Rule  
The Power Rule for the Case  $p = -5$   
The Power Rule for the Case  $p = 5/6$   
The Power Rule for the Case  $p = -4/7$   
The Power Rule for Fractional Exponents  
Optional More Careful Explanation of the Power Rule

### 3.1.4 Derivatives of Polynomials

Introducing the Idea of a Polynomial  
Finding the Derivative of a Polynomial

### 3.1.5 The Leibniz Notation for Derivatives

Motivating the Leibniz Notation for Derivatives  
Introducing the Leibniz Notation for Derivatives  
The Power Rule in Leibniz Notation  
Derivative of a Polynomial in Leibniz Notation

### 3.1.6 Exercises on Derivatives

Exercise 1:  $\frac{d}{dx} \frac{1}{\sqrt[3]{x^4}}$

Exercise 2:  $\frac{d}{dx} \frac{5}{\sqrt[4]{x^7}}$

Exercise 3:  $y = 8x^3 - 6x - 1$  tangent line problem

Exercise 4:  $y = \frac{1}{\sqrt{x}}$  tangent line problem

Exercise 5: Tangent from  $(-2, -21)$  to  $y = x^2$

Exercise 6: Tangent from  $(-3, 1)$  to  $y = \frac{1}{x}$

Exercise 7:  $f(x) = |x - 3|$  no derivative at 3

Exercise 8:  $\lim_{x \rightarrow 0} x \sin \frac{1}{x} = 0$

Exercise 9:  $x^2 \sin \frac{1}{x}$  derivative at 0?

## Document: 3.2 Elementary Facts About Derivatives

Movie:  3.2 Elementary Facts About Derivatives

### 3.2.1 The Rules for Differentiation

The Sum Rule

Stating the Sum Rule  
Explaining the Sum Rule

The Difference Rule

Stating the Difference Rule  
Explaining the Difference Rule

The Constant Multiple Rule

Stating the Constant Multiple Rule  
Explaining the Constant Multiple Rule

The Product Rule

Stating the Product Rule  
A Needed Fact About Limits  
Explaining the Product Rule

The Quotient Rule

Stating the Quotient Rule  
Explaining the Quotient Rule  
An Optional Deeper Comment About the Proof of the Quotient Rule

### 3.2.2 Exercises on the Rules for Differentiation

Exercise 1:  $f(x) = x + \frac{1}{x}$  for  $x \neq 0$

Exercise 2: Tangent line from  $(4, 4)$  to  $y = x + \frac{1}{x}$

Exercise 3:  $\frac{d}{dx} \frac{x}{1+x^2}$

Exercise 4: Horizontal tangents to  $y = \frac{x^2}{1+x^4}$

Exercise 5:  $y = \frac{x}{x^2+4}$  tangent line problem

Exercise 6:  $f(x) = (x - 3)^2 g(x)$  tangent line problem

Exercise 7:  $\frac{d}{dx} f(x)g(x)h(x)$  extended product rule

Exercise 8:  $\frac{d}{dx} (f(x))^2 = 2f(x)f'(x)$

Exercise 9: Horizontal tangents to  $y = (2 - 3x)^5(5 + 2x)^4$

### 3.2.3 Higher Order Derivatives

### 3.2.4 Exercises on Higher Order Derivatives

Exercise 1:  $f(x) = x^7$  for each  $x$ , work out  $f^{(n)}(x)$

Exercise 2:  $f(x) = \sqrt{x}$  for each  $x > 0$ , work out  $f^{(n)}(x)$

Exercise 3:  $f(x) = \frac{1}{1 + x^2}$  find  $f''(x)$

Exercise 4: Expand  $(1 + x)^7$  using derivatives

Exercise 5: Expand  $(1 + x)^p$  using derivatives

## Document: 3.3 Derivatives of the Trigonometric Functions

Movie:  3.3 Derivatives of the Trigonometric Functions

### 3.3.1 Derivatives of the Functions sin and cos

The Derivative of sin

The Derivative of cos

### 3.3.2 Derivatives of the Other Trigonometric Functions

The Derivative of tan

Finding the Derivative of tan Directly from the Definition

The Derivative of cot

Finding the Derivative of cot Directly from the Definition

The Derivative of sec

Finding the Derivative of sec Directly from the Definition

The Derivative of csc

Finding the Derivative of csc Directly from the Definition

Summary of the Trigonometric Derivatives

### 3.3.3 Exercises on Derivatives of the Trigonometric Functions

Exercise 1:  $\frac{d}{dx} \frac{\sin x}{x}$

Exercise 2:  $\frac{d}{dx} x^2 \sin x \cos x$

Exercise 3:  $\frac{d}{dx} \frac{x \sin x}{1 + x^2}$

Exercise 4: Horizontal tangents to  $y = 2 \cos^2 x + 2 \cos x - 1$

Exercise 5:  $\frac{d}{dx} ((f(x) - \sin x)^2 + (g(x) - \cos x)^2)$

## Document: 3.4 Derivative of a Composition

Movie:  3.4 Derivative of a Composition

### 3.4.1 Composition of Functions

### 3.4.2 Some Examples of Compositions

Example 1:  $f(x) = x^2$  for every number  $x$  and  $g(u) = 3 + 5u$  for every number  $u$

Example 2:  $f(x) = 1 + x^2$  for every number  $x$  and  $g(u) = u^{100}$  for every number  $u$

Example 3:  $f(x) = 2^x$  for every number  $x$  and  $g(u) = \log_2 u$  for  $u > 0$

Example 4:  $f(x) = \frac{x-2}{1-2x}$  whenever  $x \neq \frac{1}{2}$  and  $g(u) = \frac{u-3}{1-3u}$  for  $u \neq \frac{1}{3}$

### 3.4.3 Statement of the Composition Rule

### 3.4.4 Some Examples to Illustrate the Composition Rule

Example 1:  $\frac{d}{dx} (1+x^2)^{100}$

Example 2:  $\frac{d}{dx} \sin(1+x^2)$

Example 3:  $\frac{d}{dx} \sqrt{\sin x}$

### 3.4.5 Motivating the Composition Rule

### 3.4.6 Using Leibniz Notation in the Composition Rule

### 3.4.7 A Return to the Earlier Examples on the Composition

Example 1:  $\frac{d}{dx} (1+x^2)^{100}$

Example 2:  $\frac{d}{dx} \sin(1+x^2)$

Example 3:  $\frac{d}{dx} \sqrt{\sin x}$

### 3.4.8 Some Assorted Exercises on Derivatives

Exercise 1:  $\frac{d}{dx} \sqrt{\sin(1+x^2)}$

Exercise 2:  $\frac{d}{dx} (\sin x + \cos x)^{100}$

Exercise 3:  $\frac{d}{dx} \sqrt{\sin \sqrt{x}}$

Exercise 4:  $\frac{d}{dx} (\sin x + x \cos(x^3))^{100}$

Exercise 5:  $\frac{d}{dx} \frac{\sqrt{\sin(x^3)}}{\sqrt[3]{\cos(x^2)}}$

Exercise 6: Tangent to  $y = \tan x$  at  $x = \pi/4$

Exercise 7: Tangent to  $y = \sqrt{13-x^2}$  at  $(5, 1)$

Exercise 8: Finding the Angle Between Two Graphs


Exercise 9: Angle of intersection of  $y = \sin x$  and  $y = \cos x$

Note on the Final Two Exercises

Exercise 10: The Parabola Reflection Problem

Exercise 11: The Whispering Gallery Problem

## Document: 3.5 Inverse Functions

Movie:  3.5 Inverse Functions

### 3.5.1 Domain and Range of a Function

Example 1 on Domain and Range

Example 2 on Domain and Range

Example 3 on Domain and Range

Example 4 on Domain and Range

### 3.5.2 Inverse Function of a One-One Function

One-One Functions  
 Example 1 of a One-One Function  
 Example 2 of a One-One Function  
 Inverse of a One-One Function  
 Example 1 on Inverse Functions  
 Example 2 on Inverse Functions  
 Example 3 on Inverse Functions

**3.5.3 Derivative of an Inverse Function**

Introducing the Derivative of an Inverse Function  
 Example 1 of the Derivative of an Inverse Function  
 Example 2 of the Derivative of an Inverse Function

**Document: 3.6 Derivatives of Exponential and Logarithmic Functions**

**Movie:**  **3.6 Derivatives of Exponential and Logarithmic Functions**

**3.6.1 The Key to the Differentiation of an Exponential Function**

**3.6.2 Approximate Differentiation an Exponential Function with a Computer Algebra System**

Choosing a Computer Algebra System  
 Setting up Scientific Notebook  
 Approximate Evaluation of  $\frac{d}{dx} 2^x$   
 Approximate Evaluation of  $\frac{d}{dx} 3^x$

**3.6.2 Approximate Differentiation an Exponential Function with a Computer Algebra System Interactive**

Choosing a Computer Algebra System  
 Setting up Scientific Notebook  
 Approximate Evaluation of  $\frac{d}{dx} 2^x$   
 Approximate Evaluation of  $\frac{d}{dx} 3^x$

**3.6.3 Adjusting the Base of an Exponential Function: The Number e**

Preliminary Note  
 Our Objective: To Obtain  $\frac{d}{dx} a^x = 1a^x$   
 Adjusting the Base Numerically  
 Adjusting the Base Geometrically: Animation Method  
 Adjusting the Base Geometrically: Zooming Method  
 Comparing the Graphs  $y = a^x$  and  $y = \frac{d}{dx} a^x$   
 The Function exp

**3.6.3 Adjusting the Base of an Exponential Function: The Number e Interactive Form**

Preliminary Note  
 Our Objective: To Obtain  $\frac{d}{dx} a^x = 1a^x$   
 Adjusting the Base Numerically  
 Adjusting the Base Geometrically: Animation Method  
 Adjusting the Base Geometrically: Zooming Method  
 Comparing the Graphs  $y = a^x$  and  $y = \frac{d}{dx} a^x$   
 The Function exp

**3.6.4 A More Precise Approach to the Number e**

Our Main Assumption  
 Moving from Base 2 to a General Base a  
 Some Examples Involving the Exponential Function Base e

Finding  $\frac{d}{dx} a^x$  for a General Base  $a$

The Natural (Napierian) Logarithm

The Equation  $\frac{d}{dx} \log|x| = \frac{1}{x}$

Finding  $\frac{d}{dx} \log_a x$  for a General Base  $a$

### 3.6.5 Some Exercises on Derivatives of Exponential and Logarithmic Functions

Exercise 1:  $\frac{d}{dx} x \log x$

Exercise 2:  $\frac{d}{dx} \log(5x)$

Exercise 3:  $\frac{d}{dx} \log 5 = 0$

Exercise 4:  $f(x) = \log(1 + x^2)$

Exercise 5:  $\frac{d}{dx} \log|\sin x|$

Exercise 6:  $\frac{d}{dx} \log|\sec x|$

Exercise 7:  $\frac{d}{dx} \log|\sec x + \tan x|$

Exercise 8:  $\frac{d}{dx} \log|\csc x + \cot x|$

Exercise 9:  $\frac{d}{dx} (1 + x^2)^{\sin x}$

Exercise 10:  $\frac{d}{dx} \log_{(1+x^2)}(1 + x^2 + 2x^4)$

Exercise 11:  $\lim_{x \rightarrow 0} (1 + x)^{1/x}$

Exercise 12:  $\lim_{u \rightarrow \infty} (1 + \frac{1}{u})^u$

## Document: 3.7 Inverse Trigonometric Functions

Movie:  3.7 Inverse Trigonometric Functions

### 3.7.1 The Function arccos

### 3.7.2 Some Examples to Illustrate the Function arccos

The Number  $\arccos 0$

The Number  $\arccos \frac{1}{2}$

The Number  $\arccos\left(-\frac{1}{2}\right)$

The Numbers  $\arccos \frac{1}{\sqrt{2}}$  and  $\arccos\left(-\frac{1}{\sqrt{2}}\right)$

The Numbers  $\arccos\left(\frac{\sqrt{3}}{2}\right)$  and  $\arccos\left(-\frac{\sqrt{3}}{2}\right)$

The Numbers  $\arccos(.37)$  and  $\arccos(-.37)$

### 3.7.3 Some Properties of the Function arccos

Working Out  $\cos(\arccos x)$ ,  $\sin(\arccos x)$ , and  $\tan(\arccos x)$

The Derivative of the Function arccos

The Graph of the Function arccos

### 3.7.4 The Function arcsin

### 3.7.5 Some Examples to Illustrate the Function arcsin

The Numbers  $\arcsin 1$  and  $\arcsin(-1)$

The Numbers  $\arcsin \frac{1}{2}$  and  $\arcsin \left(-\frac{1}{2}\right)$

The Numbers  $\arcsin \frac{1}{\sqrt{2}}$  and  $\arcsin \left(-\frac{1}{\sqrt{2}}\right)$

### 3.7.6 Some Properties of the Function $\arcsin$

Working Out  $\sin(\arcsin x)$ ,  $\cos(\arcsin x)$ , and  $\tan(\arcsin x)$

The Derivative of the Function  $\arcsin$

The Graph of the Function  $\arcsin$

### 3.7.7 The Function $\arctan$

### 3.7.8 Some Examples to Illustrate the Function $\arctan$

The Number  $\arctan 0$

The Numbers  $\arctan 1$  and  $\arctan(-1)$

The Numbers  $\arctan \sqrt{3}$  and  $\arctan(-\sqrt{3})$

The Numbers  $\arctan \frac{1}{\sqrt{3}}$  and  $\arctan \left(-\frac{1}{\sqrt{3}}\right)$

The Limits of  $\arctan$  at  $\infty$  and at  $-\infty$

### 3.7.9 Some Properties of the Function $\arctan$

Working out  $\tan(\arctan x)$ ,  $\sec(\arctan x)$ , and  $\sin(\arctan x)$

The Identity  $\arctan x + \arctan \left(\frac{1}{x}\right) = \frac{\pi}{2}$  for  $x > 0$

Derivative of the Function  $\arctan$

The Graph of the Function  $\arctan$

### 3.7.10 The Function $\operatorname{arcsec}$

### 3.7.11 Some Examples to Illustrate the Function $\operatorname{arcsec}$

The Numbers  $\operatorname{arcsec} 1$  and  $\operatorname{arcsec}(-1)$

The Numbers  $\operatorname{arcsec} 2$  and  $\operatorname{arcsec}(-2)$

The Numbers  $\operatorname{arcsec} \sqrt{2}$  and  $\operatorname{arcsec}(-\sqrt{2})$

### 3.7.12 Some Properties of the Function $\operatorname{arcsec}$

Working Out  $\sec(\operatorname{arcsec} x)$ ,  $\tan(\operatorname{arcsec} x)$ , and  $\sin(\operatorname{arcsec} x)$

The Derivative of the Function  $\operatorname{arcsec}$

The Graph of the Function  $\operatorname{arcsec}$

### 3.7.13 Exercises on Inverse Trigonometric Functions

Exercise 1:  $\arctan(\sqrt{2} - 1) = \frac{\pi}{8}$

Exercise 2:  $\arctan(2 - \sqrt{3}) = \frac{\pi}{12}$

Exercise 3:  $\cos(2 \arcsin u) + \cos(2 \arccos u) = 0$

Exercise 4:  $\arccos(\cos \theta) = \theta$ ?

Exercise 5:  $\cos(3 \arccos x) = 4x^3 - 3x$

Exercise 6:  $\sin(4 \arccos x) = 4x(2x^2 - 1)\sqrt{1 - x^2}$

Exercise 7:  $\tan(2 \arctan x)$  defined?


Exercise 8:  $\arctan x + \arctan \left(\frac{1}{x}\right) = -\frac{\pi}{2}$  for  $x < 0$

Exercise 9:  $\arcsin(-x) = -\arcsin x$

Exercise 10:  $\arccos(-x) = \pi - \arccos x$

Exercise 11:  $\arctan \left(\frac{1 - \cos \theta}{\sin \theta}\right) + \arctan(\cot \theta) = \frac{\pi - \theta}{2}$

## Document: 3.8 Implicit Functions

**Movie:**  **3.8 Implicit Functions**

**3.8.1 Implicit 2D Graphs**

Example 1:  $x^2 + y^2 = 25$

Example 2:  $x^2y - y^2 + xy^3 = 5$

Example 3:  $(x^2 + y^2)^2 = x^2 - y^2$

Example 4:  $x^3 + y^3 - 3xy = 0$

Example 5:  $x^5 + y^5 - 3x^2y = 0$

Example 6:  $x \sin(x^2 + y^2) + y = 0$

**3.8.2 The Implicit Function Theorem**

**3.8.3 Some Exercises on Implicit Functions**

Exercise 1: Tangent to  $x^2 + y^2 = 25$  at  $(3, 4)$

Exercise 2: Slope of  $x^2y - y^2 + xy^3 = 5$  at a general point  $(x, y)$

Exercise 3: Tangent to  $x^2y - y^2 + xy^3 = 5$  at  $(2, 1)$

Exercise 4: Slope of  $(x^2 + y^2)^2 = x^2 - y^2$  at a general point  $(x, y)$

Exercise 5: Horizontal and vertical tangents to  $x^3 + y^3 - 3xy = 0$

Exercise 6: Horizontal and vertical tangents to  $x^5 + y^5 - 3x^2y = 0$

Exercise 7: Slope of  $x \sin(x^2 + y^2) + y = 0$  at a general point  $(x, y)$

**Document: 3.9 Hyperbolic Functions**

**Movie:**  **3.9 Hyperbolic Functions**

**3.9.1 Introduction to Hyperbolic Functions**

Some Preliminary Comments

The Definitions of the Hyperbolic Functions

**3.9.2 Arithmetical Properties of the Hyperbolic Functions**

Behaviour of the Hyperbolic Functions at  $0$

“Pythagorean Identities” for the Hyperbolic Functions

Replacing  $x$  by  $-x$  in the Hyperbolic Functions

Hyperbolic Function Values at a Sum or Difference

Analogues for the Hyperbolic Functions of the Trigonometric Double and Triple Angle Identities

**3.9.3 Derivatives of the Hyperbolic Functions**

The Equation  $\frac{d}{dx} \sinh x = \cosh x$

The Equation  $\frac{d}{dx} \cosh x = \sinh x$

The Equation  $\frac{d}{dx} \tanh x = \operatorname{sech}^2 x$

The Equation  $\frac{d}{dx} \operatorname{sech} x = -\operatorname{sech} x \tanh x$

**3.9.4 Inverse Functions of the Hyperbolic Functions**

The Function  $\operatorname{arcsinh} x$

Finding  $\frac{d}{dx} \operatorname{arcsinh} x$

The Function  $\operatorname{arccosh} x$

Finding  $\frac{d}{dx} \operatorname{arccosh} x$

The Function  $\operatorname{arctanh} x$

Finding  $\frac{d}{dx} \operatorname{arctanh} x$

The Function arcsech

Finding  $\frac{d}{dx} \operatorname{arcsech} x$

### 3.9.5 Some Derivatives that Involve the Hyperbolic Functions

Example 1:  $\frac{d}{dx} \arctan(\sinh x)$

Example 2:  $\frac{d}{dx} \arctan(e^x)$

Example 3:  $\frac{d}{dx} \arcsin(\operatorname{sech} x)$

Example 4:  $\frac{d}{dx} \log(\cosh x)$

Example 5:  $\frac{d}{dx} \log(\sinh x)$

Example 6:  $\frac{d}{dx} \operatorname{arcsec}(\cosh x)$

Example 7:  $\frac{d}{dx} \arccos(\operatorname{sech} x)$

Example 8:  $\frac{d}{dx} \operatorname{arccosh}(\sec x)$

Example 9:  $\frac{d}{dx} \operatorname{arctanh}(\sin x)$

## Overview of Chapter 4: Applications of the Derivative

### Document: 4.1 Monotone Functions

Movie:  4.1 Monotone Functions

#### 4.1.1 The Graph of a Function with a Positive Derivative

The Positive Derivative Principle  
Looking at the Positive Derivative Principle Intuitively  
A Note of Caution

#### 4.1.2 Increasing and Decreasing Functions

Strictly Increasing Functions  
Increasing Functions  
Strictly Decreasing Functions  
Decreasing Functions  
Monotone Functions

#### 4.1.3 More General Version of the Positive Derivative Principle

#### 4.1.4 Exercises on Monotone Functions

Exercise 1:  $f(x) = x^2 - 4x - 5$  for all  $x$

Exercise 2:  $f(x) = x^3 - 3x^2$  for all  $x$

Exercise 3:  $f(x) = |x^2 - 4x - 5|$  for all  $x$

Exercise 4:  $f(x) = |x^3 - 3x^2|$  for all  $x$

Exercise 5:  $f(x) = \left(\frac{\log x}{x}\right)^2$  for  $x > 0$

#### 4.1.5 An Application of the Positive Derivative Principle

The Inequality  $e^x > 1$  when  $x > 0$

The Inequality  $e^x > 1 + x$  when  $x > 0$

The Inequality  $e^x > 1 + x + \frac{x^2}{2}$  when  $x > 0$

The Inequality  $e^x > 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}$  when  $x > 0$

The General Case  $e^x > 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$  when  $x > 0$

#### 4.1.6 Working out Some Important Limits

The Limit  $\lim_{x \rightarrow \infty} \frac{e^x}{x}$

The Limit  $\lim_{x \rightarrow \infty} \frac{e^x}{x^5}$

The Limit  $\lim_{x \rightarrow \infty} \frac{e^x}{x^n}$

The Limit  $\lim_{x \rightarrow \infty} \frac{\log x}{x}$

The Limit  $\lim_{x \rightarrow \infty} \frac{(\log x)^{1000000}}{x}$

The Limit  $\lim_{x \rightarrow 0^+} x \log x$

The Limit  $\lim_{x \rightarrow 0^+} x(\log x)^{1000000}$

### Document: 4.2 Drawing Graphs of Functions

Movie:  4.2 Drawing Graphs of Functions

#### 4.2.1 Maxima and Minima

Definition of Maxima and Minima

Definition of Local Maxima and Minima

#### 4.2.2 Fermat's Theorem

Statement of Fermat's Theorem

Part 1: Positive derivative not at the right endpoint

Part 2: Negative derivative not at the right endpoint

Part 3: Positive derivative not at the left endpoint

Part 4: Negative derivative not at the left endpoint

Part 5: Conclusion

Using Fermat's Theorem

Critical Numbers of a Function

#### 4.2.3 Some Examples to Illustrate Fermat's Theorem

Example 1:  $f(x) = x^3$  for  $-2 \leq x \leq 2$

Example 2:  $f(x) = x^3$  for  $-2 \leq x < 2$

Example 3:  $f(x) = \begin{cases} \frac{x-1}{2} & \text{if } 0 \leq x \leq 3 \\ 2x-5 & \text{if } 3 \leq x \leq 4 \end{cases}$

#### 4.2.4 Exercises on Graphs of Functions

Exercise 1:  $f(x) = x^2 - 4x - 5$  for  $-2 \leq x \leq 6$

Exercise 2:  $f(x) = x^2 - 4x - 5$  for  $3 \leq x \leq 6$

Exercise 3:  $f(x) = |x^2 - 4x - 5|$  for  $-2 \leq x \leq 6$

Exercise 4:  $f(x) = x^3 - 3x^2$  for  $-1 \leq x \leq 4$

Exercise 5:  $f(x) = \frac{x^2}{1+x^2}$  for all  $x$

Exercise 6:  $f(x) = xe^{-x}$  for  $x \geq -1$

Exercise 7:  $f(x) = xe^{-x^2}$  for all  $x$

Exercise 8:  $f(x) = x^2e^{-x^2}$  for all  $x$

Exercise 9:  $f(x) = 3 \sin^4 x - 2 \sin^3 x$  for  $0 \leq x \leq 2\pi$

Exercise 10:  $f(x) = x(\log x)^2$  for  $0 < x \leq 2$

Exercise 11:  $f(x) = x^{2/3}(6-x)^{1/3}$  for  $-1 \leq x \leq 7$

### 4.2.5 Concavity of Graphs

The Graph of a Function with a Positive Second Derivative  
The Graph of a Function with a Negative Second Derivative  
Points of Inflection

### 4.2.6 Exercises on Concavity

- Exercise 1:  $f(x) = x^3 - 3x^2$  for all  $x$   
Exercise 2:  $f(x) = \frac{x^2}{1+x^2}$  for all  $x$   
Exercise 3:  $f(x) = xe^{-x}$  for all  $x$   
Exercise 4:  $f(x) = xe^{-x^2}$  for all  $x$   
Exercise 5:  $f(x) = x^2e^{-x^2}$  for all  $x$   
Exercise 6:  $f(x) = \log(1+x^2)$  for all  $x$   
Exercise 7:  $f(x) = (\log x)^2$  for  $x > 0$   
Exercise 8:  $f(x) = x(\log x)^2$  for  $x > 0$   
Exercise 9:  $f(x) = x(\log(x^2))^2 - 3x \log(x^2)$  for  $x \neq 0$   
Exercise 10:  $f(x) = x^{2/3}(6-x)^{1/3}$  for  $-1 \leq x \leq 7$   
Exercise 11:  $f(x) = \frac{x \log x}{1+x^2}$  for  $x > 0$   
Exercise 12: Theoretical

## Document: 4.3 Applied Maxima and Minima

Movie:  4.3 Applied Maxima and Minima

### 4.3.1 Elementary Exercises on Applied Maxima and Minima

- Exercise 1: The Chicken Coop Problem  
Exercise 2: The Box Problem  
Exercise 3: The Cylindrical Can Problem  
Exercise 4: The Rectangle in a Semicircle Problem  
Exercise 5: The Isosceles Triangle in a Parabola Problem  
Exercise 6: The Isosceles Triangle in a Circle Problem  
Exercise 7: The Cone in a Hemisphere Problem  
Exercise 8: The Triangle and Semicircle Problem  
Exercise 9: The Road and Field Problem (Special Case)  
Exercise 10: The Dimmer Switch Problem  
Exercise 11: An Electric Circuit Problem

### 4.3.2 The General Road and Field Problem (and Deriving Snell's Law)

The Narrow Road Version of the Road and Field Problem  
The Wide Road Version of the Road and Field Problem  
The Road and Field Problem and the Laws of Refraction  
Comparing the Wide Road Problem with the Narrow Road Problem

### 4.3.3 Making a Quadrilateral of Maximum Area

Maximizing the Area of a Quadrilateral with Given Sides  
The Three Sticks Problem

### 4.3.4 The Ice Cream Problem: Maximum Minimum Problems About Cones

Background Information About Cones  
Maximizing the Volume of a Cone with a Given Slant Height  
Minimizing the Slant Height of a Cone with a Given Volume  
Maximizing the Volume of a Cone with Given Surface Area  
Filling the Cone with Ice Cream

### 4.3.5 Introducing The Soapbox Car Problem (See Section 8.4 for the full discussion.)

## Document: 4.4 Antiderivatives (Indefinite Integrals)

Movie:



## 4.4 Antiderivatives (Indefinite Integrals)

### 4.4.1 Antiderivative of a Function

### 4.4.2 Some Examples of Antiderivatives

Example 1: Antiderivative with respect  $x$  of  $6x$

Example 2: Another antiderivative with respect  $x$  of  $6x$

Example 3: Antiderivative with respect  $x$  of  $\cos x$

Example 4: Antiderivative with respect  $x$  of  $\frac{1}{x}$  when  $x > 0$

Example 5: Antiderivative with respect  $x$  of  $\frac{1}{x}$  when  $x < 0$

Example 6: Antiderivative with respect  $x$  of  $\frac{1}{x}$  when  $x \neq 0$

Example 7: Antiderivative with respect  $x$  of  $x^p$  when  $p \neq -1$

### 4.4.3 The Key Fact About Antiderivatives

Statement of the Key Fact

Finding all Possible Antiderivatives of a Given Function

### 4.4.4 Some Examples of General Antiderivatives

Example 1:  $\int x dx = \frac{x^2}{2} + c$

Example 2:  $\int x^p dx = \frac{x^{p+1}}{p+1} + c$

Example 3:  $\int \frac{1}{x} dx = \log|x| + c$

Example 4:  $\int \cos x dx = \sin x + c$

Example 5:  $\int \sin x dx = -\cos x + c$

Example 6:  $\int \sec^2 x dx = \tan x + c$

Example 7:  $\int \sec x \tan x dx = \sec x + c$

Example 8:  $\int \tan x dx = \log|\sec x| + c$

Example 9:  $\int \cot x dx = \log|\sin x| + c$

Example 10:  $\int \sec x dx = \log|\sec x + \tan x| + c$

Example 11:  $\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + c$

Example 12:  $\int \frac{1}{1+x^2} dx = \arctan x + c$

Example 13:  $\int \frac{1}{x\sqrt{x^2-1}} dx = \operatorname{arcsec} x + c$

### 4.4.5 Changing Variable to Find an Antiderivative

Motivating the Change of Variable Method: Example 1

Motivating the Change of Variable Method: Example 2

Motivating the Change of Variable Method: Example 3

Motivating the Change of Variable Method: Example 4

Motivating the Change of Variable Method: Example 5

Motivating the Change of Variable Method: Example 6

Motivating the Change of Variable Method: Example 7

Introducing the Change of Variable Method

Applying The Change of Variable Method

#### 4.4.6 Some Exercises on Changing Variable

- Exercise 1:  $\int \sqrt{1+x^2} 2x dx$
- Exercise 2:  $\int \frac{4x+3}{\sqrt{2x^2+3x+7}} dx$
- Exercise 3:  $\int \frac{x}{1+x^2} dx$
- Exercise 4:  $\int \cos^4 x \sin x dx$
- Exercise 5:  $\int \sqrt{\tan x} \sec^2 x dx$
- Exercise 6:  $\int \frac{\cos(\log x)}{x} dx$
- Exercise 7:  $\int e^x \sin(3e^x) dx$
- Exercise 8:  $\int (\log \sin x)^2 \cot x dx$
- Exercise 9:  $\int x\sqrt{x+3} dx$
- Exercise 10:  $\int \sqrt{\sin x} \cos x dx$
- Exercise 11:  $\int \sqrt{\sin x} \cos^3 x dx$
- Exercise 12:  $\int \sqrt{\sin x} \cos^5 x dx$
- Exercise 13:  $\int \sqrt{\cos x} \sin^5 x dx$
- Exercise 14:  $\int \sec^6 x \sqrt{\tan x} dx$
- Exercise 15:  $\int \sec^3 x \tan^5 x dx$
- Exercise 16:  $\int (1+x) dx$  (two ways)
- Exercise 17:  $\int \sin 2\theta d\theta$  (two ways)
- Exercise 18:  $\int \frac{1}{1-x^2} dx$
- Exercise 19:  $\int \sec x dx$
- Exercise 20:  $\int \csc x dx$

#### 4.4.7 Antiderivatives that Involve Hyperbolic Functions


- Exercise 1:  $\int \cosh x dx = \sinh x + c$
- Exercise 2:  $\int \sinh x dx = \cosh x + c$
- Exercise 3:  $\int \operatorname{sech} x dx = 2 \arctan(e^x) + c$
- Exercise 4:  $\int \tanh x dx = \log \cosh x + c$
- Exercise 5:  $\int \sqrt[3]{\tanh x} \operatorname{sech}^2 x dx$
- Exercise 6:  $\int \frac{1}{\sqrt{x^2+1}} dx = \operatorname{arcsinh} x + c$
- Exercise 7:  $\int \frac{\operatorname{arcsinh} x}{\sqrt{x^2+1}} dx$
- Exercise 8:  $\int \frac{\cos x}{\sqrt{1+\sin^2 x}} dx$

Exercise 9:  $\int \frac{\sqrt{\operatorname{arccosh} x}}{\sqrt{x^2 - 1}} dx$

Exercise 10:  $\int \frac{1}{1 - x^2} dx = \operatorname{arctanh} x + c$

Exercise 11:  $\int \frac{1}{x\sqrt{1 - x^2}} dx = -\operatorname{arcsech} x + c$

## Document: 4.5 Rates of Change

Movie:  4.5 Rates of Change

### 4.5.1 Interpreting the Derivative as a Rate of Change

### 4.5.2 Some Exercises on Derivatives as Rates of Change

- Exercise 1. Inflating a Balloon: Part 1
- Exercise 2. Inflating a Balloon: Part 2
- Exercise 3. A Leaking Cone: Part 1
- Exercise 4. A Leaking Cone: Part 2
- Exercise 5. Water Evaporating from a Cone
- Exercise 6. Growth of a Bacterial Colony
- Exercise 7. Growth of Money in a Bank Account
- Exercise 8. Radioactive Decay

## Document: 4.6 Motion of a Particle in a Straight Line

Movie:  4.6 Motion of a Particle in a Straight Line

### 4.6.1 The Position Function of a Moving Particle

### 4.6.2 Examples to Illustrate Position Functions

- Example 1:  $f(t) = t^2$  for  $-1 \leq t \leq 1$
- Example 2:  $f(t) = t^4$  for  $-1 \leq t \leq 1$
- Example 3:  $f(t) = t^2$  for  $t \geq 0$
- Example 4:  $f(t) = \sin t$  for  $t \geq 0$

### 4.6.3 Velocity, Speed, and Acceleration of a Particle

### 4.6.4 Some Exercises on Velocity, Speed, and Acceleration

- Exercise 1:  $f(t) = t^2$  at each time  $t$
- Exercise 2:  $f(t) = \sin t$  at each time  $t$  in the interval  $[0, 6\pi]$
- Exercise 3:  $f'(t) = 5t$  for each time  $t$
- Exercise 4:  $f''(t) = 20$  for every  $t$

### 4.6.5 Expressing Velocity and Acceleration in Terms of Position

- An Example to Illustrate Velocity and Acceleration at a Point  $x$
- A Formula for Velocity in Terms of Position
- Returning to the Example
- A Formula for Acceleration in Terms of Position
- Returning, Once Again, to the Example

### 4.6.6 Newton's Law

- Introducing the Concept of Mass
- Introducing the Concept of Force
- Introduction to Newton's Law

- The Role of Force when Mass is Changing: The Sticky Ball Example
- The Role of Force when Velocity is Changing
- Newton's Law of Motion when the Force Acts in the Direction of the Number Line
- Newton's Law of Motion when the Force Acts Against the Direction of the Number Line
- Units to Be Used in Newton's Law
  - The Kilogram, the Newton, and the Meter
  - The Gram, the Dyne, and the centimeter
  - The Pound Mass, the Poundal, and the Foot
  - The Slug, the Pound Force, and the Foot (Included Reluctantly)

 **4.6.7 Some Exercises on Newton's Law**

- Exercise 1: A Constant Mass Propelled by a Constant Force
- Exercise 2: A Constant Mass Projected Upward Near the Ground
- Exercise 3: A Sticky Ball Coasting in a Dust Cloud
- Exercise 4: A Sticky Ball Coasting in a Resisting Dust Cloud
- Exercise 5: Another Sticky Ball Problem
- Exercise 6: A Particle Coasting in a Resisting Medium; Resistance Proportional to the Velocity
- Exercise 7: A Particle Coasting in a Resisting Medium; Resistance Proportional to the Square of the Velocity
- Exercise 8: A Rocket Problem
- Exercise 9: A Particle Moving Away from the Earth
- Exercise 10: A Relativistic Problem



## Overview of Chapter 5: The Mean Value Theorem and its Applications

 **Document: 5.1 The Mean Value Theorem**

**Movie:**  **5.1 The Mean Value Theorem**

 **5.1.1 Introduction to the Mean Value Theorem**

- Why Do We Need the Mean Value Theorem?
- A Sneak Preview of the Mean Value Theorem
- Statement of the Mean Value Theorem
- The Speeding Ticket Problem

 **5.1.2 Rolle's Theorem**

- The Statement of Rolle's Theorem
- Two Important Ingredients Needed for Rolle's theorem
  - A Brief Restatement of Fermat's theorem
  - A Brief Restatement of the Theorem on Maxima and Minima of Continuous Functions
- Proof of Rolle's Theorem
- A Two Function Version of Rolle's Theorem
- Proof of the Mean Value Theorem

 **5.1.3 Proving the Positive Derivative Principle**

- Proof of Assertion 1
- Proof of Assertion 2
- Proof of Assertion 3
- Proof of Assertion 4
- Proof of Assertion 5

 **5.1.4 Some Exercises on the Mean Value Theorem**

- Exercise 1: A function with a maximum
- Exercise 2: The derivative of a strictly increasing function
- Exercise 3: Reversing the endpoints of the interval
- Exercise 4: A condition for a function to be one-one
- Exercise 5: Using the inequality  $|f'(x)| \leq 1$

Exercise 6: When the inequality  $|f(t) - f(x)| \leq |t - x|^2$  holds

Exercise 7: A condition for two functions to be  $\sin$  and  $\cos$

Exercise 8: Derivatives have an intermediate value property

Exercise 9: A two function version of Exercise 8

## **Document: 5.2 Approximating a Function with Polynomials**

### **Movie:** **5.2 Approximating a Function with Polynomials**

#### **5.2.1 Introduction to Polynomials**

Definition of a Polynomial

Expanding  $(1 + x)^8$ : Motivating the Binomial Theorem

The Binomial Theorem

#### **5.2.2 The Coefficients of a General Polynomial**

Special Notation for Higher Derivatives of a Function

Finding the Coefficients of a Given Polynomial

The Degree of a Polynomial

Recentering the Terms of a Polynomial

#### **5.2.3 Taylor Polynomials of a Function**

Definition of The Taylor Polynomials

#### **5.2.4 Some Examples of Taylor Polynomials**

Example 1:  $f(x) = 2 - 4x + 3x^2 + 7x^3 + 5x^4$  for each  $x$

Example 2:  $f(x) = \frac{1}{1 + x^2}$  for each  $x$

Example 3:  $f(x) = \frac{1}{1 + x^2}$ , Taylor polynomials centered at  $1$

Example 4: Using a computer algebra system to find Taylor polynomials

Example 5: Another application of a computer algebra system

#### **5.2.5 Finding The Remainder Term**

Introducing the Remainder Term of a Taylor Polynomial

A Quick Review of Rolle's Theorem

A Version of Rolle's Theorem for the Second Derivative

A Version of Rolle's Theorem for the Third Derivative

A Version of Rolle's Theorem for the Fourth Derivative

Motivating the Higher Derivative Form of the Mean Value Theorem: A Mean Value Theorem for the Fourth Derivative

The Higher Derivative Form of the Mean Value Theorem (Sometimes Called the Taylor Mean Value Theorem)

#### **5.2.6 Some Applications of the Taylor Mean Value Theorem**

Finding an Approximation to  $e$

The Number  $e$  is Irrational

Finding an Approximation to  $e^3$

Finding an approximation to  $\log\left(\frac{3}{2}\right)$

Finding an approximation to  $\log\left(\frac{1}{2}\right)$

Finding An Approximation to  $\cos 1$

The Number  $\cos 1$  Is Irrational

## **Document: 5.3 Indeterminate Forms**

### **Movie:** **5.3 Indeterminate Forms**

### 5.3.1 Introduction to Indeterminate Forms

### 5.3.2 Some Examples to Illustrate Indeterminate Forms

Example 1:  $\lim_{x \rightarrow 0} \frac{3x}{x} = 3$

Example 2:  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

Example 3:  $\lim_{x \rightarrow 0^+} x(\log x) = 0$

Example 4:  $\lim_{x \rightarrow \infty} \frac{(\log x)^3}{x} = 0$

Example 5:  $\lim_{x \rightarrow 0} (1 + 2x)^{1/x} = e^2$

Example 6:  $\lim_{x \rightarrow \infty} (\sqrt{x^2 + 3x + 1} - \sqrt{x^2 - 2x + 7}) = \frac{5}{2}$

### 5.3.3 L'Hôpital's rule

Introducing L'Hôpital's rule

More Careful Statement of L'Hôpital's rule

Some Remarks About L'Hôpital's rule

The Rule Works for One-Sided and Two-Sided Limits

The Limit May Be Finite or Infinite

The Case in Which  $\lim_{x \rightarrow a} g(x) = \infty$

A Brief History of L'Hôpital's rule

A Special Case of L'Hôpital's Rule

Example 1 Showing Use of the Special Case of L'Hôpital's Rule

Example 2 Showing Use of the Special Case of L'Hôpital's Rule

Proof of the Special Case of L'Hôpital's Rule

### 5.3.4 Exercises on Indeterminate Forms

Exercise 1:  $\lim_{x \rightarrow \infty} \frac{3x - 7}{2x + 5}$

Exercise 2:  $\lim_{x \rightarrow 0} \frac{e^x - 1}{x}$

Exercise 3:  $\lim_{x \rightarrow 0} \frac{e^x \sin 5x - \sin 3x}{x}$

Exercise 4:  $\lim_{x \rightarrow 0} \left( \frac{\tan x - x}{x - \sin x} \right)$

Exercise 5:  $\lim_{x \rightarrow 0} \left( \frac{x - \sin x}{x^3} \right)$

Exercise 6:  $\lim_{x \rightarrow 0^+} x \log x$

Exercise 7:  $\lim_{x \rightarrow 1} \frac{\log x}{x - 1}$

Exercise 8:  $\lim_{x \rightarrow \infty} \frac{\log x}{x}$

Exercise 9:  $\lim_{x \rightarrow \infty} \frac{(\log x)^2}{x}$

Exercise 10:  $\lim_{x \rightarrow \infty} \frac{(\log x)^p}{x} = 0$

Exercise 11:  $\lim_{x \rightarrow \infty} (\log(3x + 2) - \log(2x - 5))$

Exercise 12:  $\lim_{x \rightarrow \infty} \frac{\log(x + 2)}{\log(x - 5)}$

Exercise 13:  $\lim_{x \rightarrow \infty} \left( \frac{(\log(3x + 2))^2 - (\log(2x - 5))^2}{\log x} \right)$

Exercise 14:  $\lim_{x \rightarrow \infty} \frac{\exp(\sqrt{\log x})}{x}$

Exercise 15:  $\lim_{x \rightarrow \pi/2} (\sin x)^{\tan x}$

Exercise 16:  $\lim_{x \rightarrow \infty} x^{(\log x)/x}$

Exercise 17:  $\lim_{x \rightarrow 0} (1 + px)^{1/x}$

Exercise 18:  $\lim_{x \rightarrow 0} \frac{e - (1 + x)^{1/x}}{x}$

Exercise 19:  $\lim_{x \rightarrow \infty} ((\log(x + 1))^2 - (\log x)^2)$

Exercise 20:  $\lim_{x \rightarrow \infty} \frac{(x + 1)^{\log(x+1)}}{x^{\log x}}$

Exercise 21:  $\lim_{x \rightarrow \infty} \left( \frac{\log(x + 1)}{\log x} \right)^x$

Exercise 22:  $\lim_{x \rightarrow \infty} \frac{x e^{\sin x}}{\log x}$

5.3.5 An Important Limit:  $\lim_{x \rightarrow \infty} x \left( 1 - \frac{x^p}{(x + 1)^p} \right)$

## Overview of Chapter 6: Integrals

### Document: 6.1 Introducing Integrals as Antiderivatives

Movie:  6.1 Introducing Integrals as Antiderivatives

#### 6.1.1 Preliminary Note: This Movie Takes the Fast Track into Integral Calculus

#### 6.1.2 Defining Integrals Using Antiderivatives

Reviewing a Property of Antiderivatives

Defining the Symbol  $\int_a^b f(x) dx$

Notation for Taking a Function Between Limits

The Symbol  $x$  is Not Important

#### 6.1.3 Some Examples to Illustrate the Definition of an Integral

Example 1:  $\int_2^5 x dx$

Example 2:  $\int_0^{\pi/2} \cos x dx$

Example 3:  $\int_2^9 \frac{1}{x} dx$

Example 4:  $\int_0^{\pi/4} \sec^2 x dx$

Example 5:  $\int_0^{\pi/4} \sec x \tan x dx$

Example 6:  $\int_0^{\pi/4} \sec x dx$

Example 7:  $\int_{-1}^2 2x\sqrt{1+x^2} dx$

#### 6.1.4 Linearity and Additivity of the Integral

Linearity of the Integral

Additivity of the Integral

The Symbol  $\int_b^a$  when  $a < b$

#### 6.1.5 Using Integrals to Find Area

The Area Under the Graph of a Nonnegative Function: Historical Approach Using Infinitesimals

The Area Under the Graph of a Nonnegative Function Without Using Infinitesimals

Area of the Region Between Two Graphs  
 Area Between the Graph of a Negative Function and the  $x$ -Axis

**6.1.6 Some Exercises on Area**

- Exercise 1: The region between  $y = 4 - x^2$  and the  $x$ -axis
- Exercise 2: A triangular region
- Exercise 3: Region between  $y = x^3 - 3x^2 + 2$  and  $y = -x^2 + 3x + 2$
- Exercise 4: Region between  $y = \sin x$  and  $y = \cos x$
- Exercise 5: Region between  $y = \sin x$  and  $y = \sin 2x$
- Exercise 6: Region between  $y = \sin x$  and  $y = \sqrt{\sin x} \cos x$

**6.1.7 Derivatives of Integrals: The Equation  $\frac{d}{dx} \int_a^x f(t) dt = f(x)$**

**6.1.8 Exercises on Derivatives of Integrals**

- Exercise 1:  $\frac{d}{dx} \int_1^x \sqrt{1+t+t^4} dt$
- Exercise 2:  $\frac{d}{dx} \int_1^x \sqrt{1+t+t^4} dt$
- Exercise 3:  $\frac{d}{dx} \int_2^{\sin x} \sqrt{1+t+t^4} dt$
- Exercise 4:  $\frac{d}{dx} \int_2^{\log x} \sqrt[3]{1+\sin^2 t} dt$
- Exercise 5:  $\frac{d}{dx} \int_{\exp(\sin x)}^5 \sqrt[3]{1+t^2} dt$
- Exercise 6:  $\frac{d}{dx} \int_{\sin x}^{\exp(x^2)} \sqrt{1+t^4} dt$

**Document: 6.2 Riemann Sums**

**Movie:**  **6.2 Riemann Sums**

**6.2.1 Summation Notation**

Introducing Summation Notation  
 Some Simple Examples to Illustrate Summation Notation

Example 1:  $\sum_{j=3}^5 j^3$

Example 2:  $\sum_{j=0}^7 (-1)^j$

Example 3:  $\sum_{j=1}^n 4$

Arithmetical Rules for Summation

Working Out the Sum  $\sum_{j=1}^n j$

Another Way of Working Out  $\sum_{j=1}^n j$

Working Out the Sum  $\sum_{j=1}^n j^2$

Working Out the Sum  $\sum_{j=1}^n j^3$

Using a Computer Algebra System to Work Out  $\sum_{j=1}^n j^p$

## 6.2.2 Introduction to Riemann Sums

- Motivating Riemann Sums
- Definition of a Partition
- Definition of a Riemann Sum
- Regular Partitions
- Darboux's Theorem
- Left Sums, Right Sums, and Midpoint Sums
  - Left Sums
  - Right Sums
  - Midpoint Sums

## 6.2.3 Some Examples to Illustrate Darboux's Theorem

Example 1:  $\int_0^1 x dx$

Example 2:  $\int_0^1 x^2 dx$

Example 3:  $\int_a^b x^2 dx$

Example 4:  $\int_0^1 \sqrt{x} dx$

Example 5:  $\int_0^1 \sqrt[3]{x^2} dx$

## Document: 6.3 Riemann Sums with a Computer Algebra System

Movie:  6.3 Riemann Sums with a Computer Algebra System

### 6.3.1 Introductory Comment

### 6.3.2 Setting up The Riemann Sums

- Supplying the Regular Partition to a Computer Algebra System
- Introducing a Temporary Function  $f$
- Defining the Left Sum of a Function
- Defining the Right Sum of a Function
- Defining the Trapezoidal Sum of a Function
- Defining Midpoint Sum of a Function
- Defining the Simpson Sum of a Function
- Motivation of the Simpson Sum

### 6.3.3 Numerical Approximations to Integrals

Summary of the Definitions

Using the Sums to Estimate  $\int_0^1 \sqrt[3]{1+x^2} dx$

Using the Sums to Estimate  $\int_0^1 \sqrt[3]{1-x^2} dx$

Obtaining Arrays of Approximating Sums Automatically

## Document: 6.4 Using Riemann Sums to Define an Integral

Movie:  6.4 Using Riemann Sums to Define an Integral

### 6.4.1 Our Objective in this Section

### 6.4.2 A Quick Review of Riemann Sums

- Bounded Functions
- Definition of a Partition

Definition of a Riemann Sum  
Regular Partitions

### 6.4.3 Squeezing a Function, Integrability, and the Integral

Motivating the Idea of Squeezing  
Definition of a Squeezing Pair of Sequences  
A Key Fact About a Squeezing Pair of Sequences  
Integrability and the Integral

### 6.4.4 Some Examples to Illustrate Integrability

Example 1: The Integral  $\int_0^1 x dx$

Example 2: The Integral  $\int_0^1 x^2 dx$

Example 3: Increasing Functions Are Integrable  
Example 4: Decreasing Functions Are Integrable  
Example 5: Continuous Functions Are Integrable  
Example 6: A Function that Fails to be Integrable

### 6.4.5 Some Facts About the Integral

Linearity of the Integral  
Nonnegativity of the integral  
Additivity of the Integral  
Darboux's Theorem

### 6.4.6 The Fundamental Theorem of Calculus

Part 1 of the Fundamental Theorem of Calculus  
Part 2 of the Fundamental Theorem of Calculus

### 6.4.7 Optional Item: Error Estimates for the Simpson Sum (Not included in the video)

Background for the Error Estimates  
An Example to Illustrate the Error Estimates

## Overview of Chapter 7: Evaluating Integrals

### Document: 7.1 Evaluating Integrals by Substitution

Movie:  7.1 Evaluating Integrals by Substitution

### 7.1.1 Some Common Antiderivatives

The Antiderivative  $\int x^p dx$  when  $p \neq -1$

The Antiderivative  $\int e^x dx$

The Antiderivative  $\int x^p dx$  when  $p = -1$

The Antiderivative  $\int \cos x dx$

The Antiderivative  $\int \sin x dx$

The Antiderivative  $\int \sec^2 x dx$

The Antiderivative  $\int \sec x \tan x dx$

The Antiderivative  $\int \tan x dx$

The Antiderivative  $\int \cot x dx$

The Antiderivative  $\int \sec x dx$

The Antiderivative  $\int \frac{1}{\sqrt{1-x^2}} dx$

The Antiderivative  $\int \frac{1}{1+x^2} dx$

The Antiderivative  $\int \frac{1}{x\sqrt{x^2-1}} dx$

## The List of Antiderivatives

### 7.1.2 Changing Variable to Calculate an Integral

Introducing the Change of Variable Method  
Applying the Change of Variable Method

### 7.1.3 Some Exercises on the Change of Variable Method

Exercise 1:  $\int_0^{\pi/2} \sin^2 x \cos x dx$

Exercise 2:  $\int_0^1 \sqrt{1+x^2} 2x dx$

Exercise 3:  $\int_0^1 \frac{4x+3}{\sqrt{2x^2+3x+7}} dx$

Exercise 4:  $\int_1^2 \frac{x}{1+x^2} dx$

Exercise 5:  $\int_0^{\pi} \cos^4 x \sin x dx$

Exercise 6:  $\int_0^{\pi/4} \sqrt{\tan x} \sec^2 x dx$

Exercise 7:  $\int_0^{\pi/3} \tan^2 x dx$

Exercise 8:  $\int_0^{\pi/3} \tan^3 x dx$

Exercise 9:  $\int_0^{\pi/4} \tan^4 x dx$

Exercise 10:  $\int_1^{\exp(\pi/3)} \frac{\cos(\log x)}{x} dx$

Exercise 11:  $\int_{\log(\pi/12)}^{\log(\pi/6)} e^x \sin(3e^x) dx$

Exercise 12:  $\int_0^1 x \sqrt{x+3} dx$

Exercise 13:  $\int_0^{\pi/2} \sqrt{\sin x} \cos x dx$

Exercise 14:  $\int_0^{\pi/2} \sqrt{\sin x} \cos^3 x dx$

Exercise 15:  $\int_0^{\pi/2} \sqrt{\sin x} \cos^5 x dx$

Exercise 16:  $\int_0^{\pi/2} \sqrt{\cos x} \sin^5 x dx$

Exercise 17:  $\int_0^{\pi/2} \cos^2 x dx$

Exercise 18:  $\int_0^{\pi/2} \sin^4 x \cos^2 x dx$

Exercise 19:  $\int_0^{\pi} \sqrt[3]{1+2\sin^2 x + \sin^5 x} \cos x dx$

Exercise 20:  $\int_0^{\pi/4} \sec^6 x \sqrt{\tan x} dx$

Exercise 21:  $\int_0^{\pi/3} \sec^3 x \tan^5 x dx$

Exercise 22:  $\int_0^{\pi/3} \sec x dx$

Exercise 23:  $\int_0^{1/2} \frac{\sqrt[3]{\arcsin x}}{\sqrt{1-x^2}} dx$

Exercise 24:  $\int_1^{\sqrt{3}} \frac{1}{(1+x^2) \arctan x} dx$

Exercise 25:  $\int_0^1 \frac{\arctan x}{(1+x^2) \sqrt{1+(\arctan x)^2}} dx$

Exercise 26:  $\int_{\sqrt{2}}^2 \frac{1}{x\sqrt{x^2-1} \operatorname{arcsec} x} dx$

## Document: 7.2 Evaluating Integrals by Parts

Movie:  7.2 Evaluating Integrals by Parts

### 7.2.1 Introduction to Integration by Parts

### 7.2.2 Some Examples to Illustrate Integration by Parts

Example 1:  $\int_0^{\pi/2} x \cos x dx$

Example 2:  $\int_0^1 x e^{3x} dx$

Example 3:  $\int_0^{\pi/2} \cos^2 x dx$

### 7.2.3 Explaining Integration by Parts

Explaining Integration by Parts for Integrals  
Explaining Integration by Parts for Antiderivatives

### 7.2.4 Exercises on Integration by Parts

Exercise 1

Exercise 1 Part a:  $\int_0^{\pi/2} x^2 \cos x dx$

Exercise 1 Part b:  $\int x^2 \cos x dx$

Exercise 2

Exercise 2 Part a:  $\int_1^2 x \log x dx$

Exercise 2 Part b:  $\int x \log x dx$

Exercise 3

Exercise 3 Part a:  $\int_1^2 x(\log x)^2 dx$

Exercise 3 Part b:  $\int x(\log x)^2 dx$

Exercise 4

Exercise 4 Part a:  $\int_1^2 x(\log x)^3 dx$

Exercise 4 Part b:  $\int x(\log x)^3 dx$

Exercise 5:  $\int_1^2 \log x dx$

Exercise 6:  $\int_0^{\pi^2/4} \cos \sqrt{x} dx$

Exercise 7:  $\int_0^1 \arctan x dx$

Exercise 8:  $\int_0^1 x \arctan x dx$

Exercise 9:  $\int_0^{1/2} \arcsin x dx$

Exercise 10:  $\int_0^{\pi/2} x \sin x \cos x dx$

Exercise 11:  $\int_0^1 x \arcsin x dx$

Exercise 12:  $\int_0^{\pi} e^x \cos x dx$

Exercise 13:  $\int_0^{\pi/3} \sec^3 x dx$

Exercise 14:  $\int_0^{\log \sqrt{3}} \operatorname{sech}^3 x dx$

Exercise 15:  $\int_0^{2\pi} \cos mx \cos n x dx$

### 7.2.5 Reduction Formulas

Introduction to Reduction Formulas

Example 1: A Reduction Formula for the Integral  $\int_1^2 x(\log x)^n dx$

Example 2: A Reduction Formula for the Antiderivative  $\int x(\log x)^n dx$

Example 3: A Reduction Formula for the Integral  $\int_0^{\pi/2} x^n \cos x dx$

Example 4: A Reduction Formula for the Antiderivative  $\int x^n e^x dx$

Example 5: A Reduction Formula for the Antiderivative  $\int \cos^n x dx$

Example 6: A Reduction Formula for the Integral  $\int_0^{\pi/2} \cos^n x dx$

Example 7: A Reduction Formula for the Antiderivative  $\int \sin^n x dx$

Example 8: A Reduction Formula for the Integral  $\int_0^{\pi/2} \sin^n x dx$

Example 9: A Reduction Formula for the Antiderivative  $\int \tan^n x dx$

Example 10: A Reduction Formula for the Antiderivative  $\int \cot^n x dx$

Example 11: A Reduction Formula for the Antiderivative  $\int \sec^n x dx$

Example 12: A Reduction Formula for the Integral  $\int_0^{\pi/4} \sec^n x dx$

### 7.2.6 Wallis' Formula: $\lim_{n \rightarrow \infty} \frac{2^{2n} (n!)^2}{\sqrt{n} (2n)!} = \sqrt{\pi}$

Introduction to Wallis' Formula

A Return to the Integral  $\int_0^{\pi/2} \cos^n x dx$

Deriving Wallis' Formula

### Document:

## 7.3 Evaluating Integrals Using Trigonometric and Hyperbolic Substitutions

### Movie Option 1:



### 7.3 Evaluating Integrals Using Trigonometric Substitutions Only

### Movie Option 2:



### 7.3 Evaluating Integrals Using Trigonometric and Hyperbolic Substitutions

#### 7.3.1 Preliminary Notes

Introduction to this Section

How Do I Know Whether to Use Trig or Hyperbolic Substitutions?

How Do I Know Whether a Given Integral Is of Type 1, 2, or 3?

#### 7.3.2 Substitutions Involving $\sin$ or $\tanh$

Introduction to the  $\sin$  Substitution

An Example to Illustrate the  $\sin$  Substitution

Introduction to the  $\tanh$  Substitution  
 An Example to Illustrate the  $\tanh$  Substitution  
 Integrals of Expressions Involving  $\sqrt{a^2 - x^2}$

### 7.3.3 Substitutions Involving $\sec$ or $\cosh$

Introduction to the  $\sec$  Substitution  
 An Example to Illustrate the  $\sec$  Substitution  
 Introduction to the  $\cosh$  Substitution  
 An Example to Illustrate the  $\cosh$  Substitution  
 Integrals of Expressions Involving  $\sqrt{x^2 - a^2}$

### 7.3.4 Substitutions Involving $\tan$ or $\sinh$

Introduction to the  $\tan$  Substitution  
 An Example to Illustrate the  $\tan$  Substitution  
 Introduction to the  $\sinh$  Substitution  
 An Example to Illustrate the  $\sinh$  Substitution  
 Integrals of Expressions Involving  $\sqrt{a^2 + x^2}$

### 7.3.5 Exercises on Trigonometric and Hyperbolic Substitutions

- Exercise 1:  $\int_0^{3/2} \sqrt{9 - x^2} dx$   
 Evaluation Using a Trigonometric Substitution  
 Evaluation Using a Hyperbolic Substitution
- Exercise 2:  $\int_0^3 \sqrt{9 - x^2} dx$   
 Evaluation Using a Trigonometric Substitution  
 Evaluation Using a Hyperbolic Substitution: **Omitted**
- Exercise 3:  $\int_0^{5/2} \frac{1}{\sqrt{25 - x^2}} dx$   
 Evaluation Using a Trigonometric Substitution  
 Evaluation Using a Hyperbolic Substitution
- Exercise 4:  $\int_0^3 \frac{1}{9 + x^2} dx$   
 Evaluation Using a Trigonometric Substitution  
 Evaluation Using a Hyperbolic Substitution
- Exercise 5:  $\int_{\sqrt{2}}^2 \frac{x^2}{\sqrt{x^2 - 1}} dx$   
 Evaluation Using a Trigonometric Substitution  
 Evaluation Using a Hyperbolic Substitution
- Exercise 6:  $\int_{3\sqrt{2}}^6 \frac{1}{(x^2 - 9)^{3/2}} dx$   
 Evaluation Using a Trigonometric Substitution  
 Evaluation Using a Hyperbolic Substitution
- Exercise 7:  $\int_0^1 \frac{x}{(1 + x^2)^{3/2}} dx$
- Exercise 8:  $\int_0^{1/2} \frac{x^2}{\sqrt{1 - x^2}} dx$   
 Evaluation Using a Trigonometric Substitution  
 Evaluation Using a Hyperbolic Substitution
- Exercise 9:  $\int_0^{1/2} \frac{x}{\sqrt{1 - x^2}} dx$
- Exercise 10:  $\int_{1/2}^{1/\sqrt{2}} \frac{1}{x\sqrt{1 - x^2}} dx$   
 Evaluation Using a Trigonometric Substitution  
 Evaluation Using a Hyperbolic Substitution
- Exercise 11:  $\int_0^1 \frac{x^2}{(1 + x^2)^{3/2}} dx$   
 Evaluation Using a Trigonometric Substitution

Evaluation Using a Hyperbolic Substitution

Exercise 12:  $\int_{3\sqrt{2}}^6 \frac{1}{x\sqrt{x^2-9}} dx$

Evaluation Using a Trigonometric Substitution

Evaluation Using a Hyperbolic Substitution

Exercise 13:  $\int_{3\sqrt{2}}^6 \frac{1}{x^2\sqrt{x^2-9}} dx$

Evaluation Using a Trigonometric Substitution

Evaluation Using a Hyperbolic Substitution

Exercise 14:  $\int_{3\sqrt{2}}^6 \frac{1}{x^4\sqrt{x^2-9}} dx$

Evaluation Using a Trigonometric Substitution

Evaluation Using a Hyperbolic Substitution

Exercise 15:  $\int_{3\sqrt{2}}^6 \frac{1}{\sqrt{x^2-9}} dx$

Evaluation Using a Trigonometric Substitution

Evaluation Using a Hyperbolic Substitution

Exercise 16:  $\int_{3\sqrt{2}}^6 \frac{x}{\sqrt{x^2-9}} dx$

Exercise 17:  $\int_{3\sqrt{2}}^6 \frac{x^3}{\sqrt{x^2-9}} dx$

Exercise 18:  $\int_0^{\pi/2} \frac{\cos x}{\sqrt{1+\sin^2 x}} dx$

Evaluation Using a Trigonometric Substitution

Evaluation Using a Hyperbolic Substitution

Exercise 19:  $\int_1^2 \frac{\sqrt{x^2-1}}{x^4} dx$

Evaluation Using a Trigonometric Substitution

Evaluation Using a Hyperbolic Substitution

Exercise 20:  $\int_5^{3+2\sqrt{3}} \frac{1}{\sqrt{x^2-6x+13}} dx$

Evaluation Using a Trigonometric Substitution

Evaluation Using a Hyperbolic Substitution

Exercise 21:  $\int_{1/2}^2 \frac{1}{(2x^2-2x+5)^{3/2}} dx$

Evaluation Using a Trigonometric Substitution

Evaluation Using a Hyperbolic Substitution

Exercise 22:  $\int_{1+3\sqrt{2}}^7 \frac{1}{\sqrt{x^2-2x-8}} dx$

Evaluation Using a Trigonometric Substitution

Evaluation Using a Hyperbolic Substitution

Exercise 23:  $\int_3^5 \sqrt{6x-5-x^2} dx$

Evaluation Using a Trigonometric Substitution

Evaluation Using a Hyperbolic Substitution: **Omitted**

Exercise 24:  $\int_1^{4/3} \frac{1}{(18x-9x^2-5)^{3/2}} dx$

Evaluation Using a Trigonometric Substitution

Evaluation Using a Hyperbolic Substitution

 **Document: 7.4 Integration of Rational Functions**

**Movie:**  **7.4 Integration of Rational Functions**

 **7.4.1 Background on Rational Functions**

Introducing Rational Functions

**7.4.2 Some Exercises on Integration of Rational Functions**

Exercise 1:  $\int \frac{x+23}{x^2-3x-10} dx$

Exercise 2:  $\int_0^1 \frac{3x^2+8x+7}{(x+1)(x+2)^2} dx$

Exercise 3:  $\int_{-1}^1 \frac{x^2+x+2}{(x+3)(x^2+2x+5)} dx$

Exercise 4:  $\int_{-1}^1 \frac{2x-2}{(x+3)(x^2+2x+5)} dx$

Exercise 5:  $\int_{-1}^1 \frac{x^2+5x-2}{(x+3)(x^2+2x+5)} dx$

Exercise 6:  $\int_0^{\pi/4} \sqrt{\tan x} dx$

Exercise 7:  $\int_0^{\pi/4} \sqrt[3]{\tan x} dx$

**7.4.3 Integrating Rational Functions of cos and sin**

**7.4.4 Exercises on Rational Functions of cos θ and sin θ**

Exercise 1:  $\int_0^{\pi/2} \frac{1}{\sin \theta + \cos \theta} d\theta$

Exercise 2: An Alternative Approach to  $\int_0^{\pi/2} \frac{1}{\cos \theta + \sin \theta} d\theta$

Exercise 3:  $\int_0^{\pi/2} \frac{\sin \theta}{\sin \theta + \cos \theta} d\theta$

Exercise 4:  $\int_0^{\pi/2} \frac{\sin \theta}{1 + \cos \theta + \sin \theta} d\theta$

**Document: 7.5 Evaluating Improper Integrals**

**Movie:**  **7.5 Evaluating Improper Integrals**

**7.5.1 Introduction to Improper Integrals**

Example 1: Motivating The integral  $\int_0^1 \frac{1}{\sqrt{x}} dx$

Example 2: Motivating The integral  $\int_0^1 \frac{1}{\sqrt{1-x^2}} dx$

Example 3: Motivating The integral  $\int_1^\infty \frac{1}{x^2} dx$

Definition of an Improper Integral

Definition of an Integral that Improper at its Right Endpoint

Definition of an Integral that Improper at its Left Endpoint

Convergence and Divergence of an Improper Integral

**7.5.2 Some Examples of Improper Integrals**

Example 1:  $\int_0^1 \frac{1}{\sqrt{x}} dx$

Example 2:  $\int_1^\infty \frac{1}{\sqrt{x}} dx$

Example 3:  $\int_1^\infty \frac{1}{x^2} dx$

Example 4:  $\int_0^{\infty} \cos x dx$

Example 5:  $\int_0^1 \arcsin x dx$

### 7.5.3 Some Exercises on Improper Integrals

Exercise 1:  $\int_0^{\infty} \frac{1}{(1+x^2)^{3/2}} dx$

The More Careful Approach  
The Quick Approach

Exercise 2:  $\int_2^{\infty} \frac{1}{x\sqrt{x^2-1}} dx$

The More Careful Approach  
The Quick Approach

Exercise 3:  $\int_1^2 \frac{1}{x\sqrt{x^2-1}} dx$

The More Careful Approach  
The Quick Approach

Exercise 4:  $\int_0^{\pi/2} \tan x dx$

The More Careful Approach  
The Quick Approach

Exercise 5:  $\int_0^{\pi/2} \sqrt{\tan x \sin x} dx$

The More Careful Approach  
The Quick Approach

Exercise 6:  $\int_0^{\pi/2} \frac{x \cos x - \sin x}{x^2} dx$

The More Careful Approach  
The Quick Approach: **Omitted**

Exercise 7:  $\int_0^{\infty} e^{-x} \sin x dx$

The More Careful Approach  
The Quick Approach

Exercise 8:  $\int_0^1 \frac{1}{x^p} dx$

Exercise 9:  $\int_1^{\infty} \frac{1}{x^p} dx$

Exercise 10:  $\int_2^{\infty} \frac{1}{x(\log x)^p} dx$

Exercise 11:  $\int_0^2 \frac{1}{(x-1)^{1/3}} dx$

Exercise 12:  $\int_0^1 \log x dx$

### Document: 7.6 Convergence of Improper Integrals

Movie:  7.6 Convergence of Improper Integrals

#### 7.6.1 Introduction to This Section

#### 7.6.2 Convergence of Integrals of Nonnegative Functions

An Fundamental Principle About Integrals of Nonnegative Functions

Warning

An Example to Illustrate The Fundamental Principle

Introducing The Comparison Test for Improper Integrals

The Comparison Test for Improper Integrals

An Example to Illustrate the Comparison Test  
 A Second Example to Illustrate the Comparison Test  
 Introduction to the Limit Version of the Comparison Test  
 Statement of the Limit Comparison Test  
 Another Way of Looking at the Limit Comparison Test

### 7.6.3 Exercises on the Comparison Test

Exercise 1:  $\int_1^{\infty} \frac{x}{x^3 - 3x^2 + 3x + 7} dx$

Exercise 2:  $\int_0^1 \frac{1}{\sqrt{x} \cos x} dx$

Exercise 3:  $\int_1^{\infty} \frac{\log x}{x^2} dx$

Exercise 4:  $\int_1^{\infty} \frac{1}{\sqrt[3]{x^2 + 5x + 2}} dx$

Exercise 5:  $\int_0^1 \frac{\sin^2 x}{x^{5/2}} dx$

Exercise 6:  $\int_1^{\infty} \frac{\sqrt{x}}{x^2 - x + 1} dx$

Exercise 7:  $\int_0^{\pi/2} \sqrt{\tan x} dx$

Exercise 8:  $\int_1^2 \frac{1}{\log x} dx$

Exercise 9:  $\int_0^{\pi/2} \log(\sin x) dx$

Exercise 10:  $\int_0^{\infty} x^{\alpha-1} e^{-x} dx$

Exercise 11:  $\int_0^1 x^{\alpha-1} e^{-x} dx$

**Note On the Final Three Exercises of this Group**

Exercise 12:  $\int_2^{\infty} \frac{1}{(\log x)^{\log x}} dx$

Exercise 13:  $\int_3^{\infty} \frac{1}{(\log \log x)^{\log x}} dx$

Exercise 14:  $\int_3^{\infty} \frac{1}{(\log x)^{\log \log x}} dx$

### 7.6.4 Improper Integrals of Functions that Can Change Sign

Absolute Convergence of an Improper Integral  
 Every Absolutely Convergent Integral Must Converge  
 Conditional Convergence of an Improper Integral

### 7.6.5 Exercises on Absolute and Conditional Convergence of Improper Integrals

Exercise 1:  $\int_1^{\infty} \frac{\sin x}{x^2} dx$  and  $\int_1^{\infty} \frac{\cos x}{x^2} dx$

Exercise 2:  $\int_1^{\infty} \frac{\sin x}{x} dx$

Exercise 3:  $\int_1^{\infty} \frac{\sin cx}{x^p} dx$

Exercise 4:  $\int_1^{\infty} \frac{\cos cx}{x^p} dx$

Exercise 5:  $\int_1^{\infty} \frac{\sin^2 x}{x} dx$

Exercise 6:  $\int_1^{\infty} \frac{|\sin x|}{x} dx$

Exercise 7: Conditional Convergence of  $\int_1^{\infty} \frac{\sin x}{x} dx$



## Overview of Chapter 8: Some Applications of Derivatives and Integrals

### **Document: 8.1 Using Integrals to Find Volume**

**Movie:**  **8.1 Using Integrals to Find Volume**

#### **8.1.1 Volume by the Method of Slicing**

#### **8.1.2 Exercises on the Method of Slicing**

- Exercise 1: Volume of a Cone
- Exercise 2: Volume of a Pyramid
- Exercise 3: Volume of a Ball
- Exercise 4: A Variation on the Cone Problem
- Exercise 5: Rotating a Plane Region Around the  $x$ -Axis
- Exercise 6: A Specific Region Rotated Around the  $x$ -Axis
- Exercise 7: A Return to the Volume of a Ball Exercise
- Exercise 8: Volume of a Bagel
- Exercise 9: An Apple Without its Core
- Exercise 10: Rotating a region bounded by  $y = \sin x$  and  $y = \cos x$  about the  $x$ -axis

#### **8.1.3 Volume by the Method of Shells**

- Introducing the Shell Method
- Finding the Volume of a Cylindrical Shell
- Returning to Our Introduction

#### **8.1.4 Exercises on the Method of Shells**

- Exercise 1
  - Using the Slicing Method to Find this Volume
  - Using the Shell Method to Find this Volume
- Exercise 2
  - Using the Slicing Method to Find this Volume
  - Using the Shell Method to Find this Volume
- Exercise 3
- Exercise 4
  - Using the Slicing Method to Find this Volume
  - Using the Shell Method to Find this Volume
- Exercise 5
  - Using the Slicing Method to Find this Volume
  - Using the Shell Method to Find this Volume
- Exercise 6: Using the Shell Method to Find the Volume of a Bagel

### **Document: 8.2 Work Done by a Force**

**Movie:**  **8.2 Work Done by a Force**

#### **8.2.1 Work Done by a Constant Force**

- Introducing the Units of Work
- Lifting a Mass Near the Surface of the Earth

#### **8.2.2 Work Done by a Variable Force**

- Introducing the Formula for Work Done by a Variable Force

#### **8.2.3 Exercises on Work Done by a Force**

- Exercise 1: Stretching a Piece of Elastic
- Exercise 2: Lifting a Leaking Bag of Flour
- Exercise 3: A Crane Lifting a Leaky Bag of Sand
- Exercise 4: Lifting a Constant Mass from the Ground to a Specified Distance from the Earth

### **8.2.4 Work Done by a Force Acting on a Moving Particle**

Review of the Discussion of Velocity and Acceleration in Terms of Position  
Work Done by a Force Acting on a Moving Particle

### **8.2.5 Exercises on Work Done by a Force Acting on a Particle**

- Exercise 1: Kinetic Energy of a Particle with Constant Mass
- Exercise 2: Projecting a Particle from the Earth
- Exercise 3: A Relativistic Formula for Kinetic Energy  
Einstein's Mass-Energy Relationship

## **Document: 8.3 Parametric and Polar Curves**

**Movie:**  **8.3 Parametric and Polar Curves**

### **8.3.1 Parametric Curves**

Motivating the Idea of a Parametric Curve  
Definition of a 2D Parametric Curve

### **8.3.2 Some Examples of Parametric Curves**

- Example 1: A Curve that Runs in a Parabola
- Example 2: A Restricted Form of the Curve in Example 1
- Example 3: Moving Through the Parabola Several Times
- Example 4: A Curve with a Loop
- Example 5: A Fish Curve
- Example 6: A Particle Travelling Counter Clockwise in a Circle
- Example 7: A Particle Travelling Clockwise in a Circle
- Example 8: A Spiral Curve
- Example 9: An Exponential Spiral Curve
- Example 10: The Cycloid

### **8.3.3 Distance Travelled along a Curve**

### **8.3.4 Exercises on Curve Length**

- Exercise 1: Length of a Circle
- Exercise 2: Going Twice Around a Circle
- Exercise 3: Length of a Spiral Curve
- Exercise 4: Length of an Exponential Spiral Curve
- Exercise 5: Length of a Cycloid
- Exercise 6: Length of an Ellipse

### **8.3.5 Area of a Surface of Revolution**

### **8.3.6 Exercises on Surface of Revolution**

- Exercise 1: Area of a Sphere
- Exercise 2: Area of a Cone
- Exercise 3: Area of a Paraboloid
- Exercise 4: Rotating the Graph of  $\sin$
- Exercise 5: Area of a Circular Ellipsoid

### **8.3.7 Polar Coordinates**

Introduction to Polar Coordinates  
Polar Coordinates are not Unique  
A Relationship Between Polar Coordinates and Rectangular Coordinates  
Existence of Polar Coordinates of Any Given Point

## Polar Graphs

### 8.3.8 Exercises on Polar Coordinates

- Exercise 1: Finding a Point with Given Polar Coordinates
- Exercise 2: Finding Polar Coordinates of a Given Point
- Exercise 3: Polar Equation of a Circle
- Exercise 4: Polar Equation of a Vertical Line
- Exercise 5: Polar Equation of a Horizontal Line
- Exercise 6: Polar Equation of a Line Through the Origin
- Exercise 7: Polar Equation of a Parabola
- Exercise 8: Polar Equation of a Circle with Center at  $(1, 0)$
- Exercise 9: Polar Equation of a Spiral Graph
- Exercise 10: The Polar Graph  $r = \frac{1}{\theta}$
- Exercise 11: The Polar Graph  $r = \frac{1}{\sqrt{\theta}}$
- Exercise 12: The Polar Graph  $r = \cos 2\theta$
- Exercise 13: The Polar Graph  $r = \sin 3\theta$
- Exercise 14: The Polar Graph  $r = \cos 3\theta$
- Exercise 15: The Polar Graph  $r = 1 + \cos \theta$
- Exercise 16: The Polar Graph  $r = 1 + 2 \cos \theta$
- Exercise 17: A Computer Generated Polar Graph

### 8.3.9 Length of a Polar Graph

- Introducing the Formula for Length of a Polar Graph
- Example 1: Length of a Petal of the Graph  $r = \cos 3\theta$ .
- Example 2: Length of a Cardioid
- Example 3: Length of a Limacon

### 8.3.10 Area Bounded by a Polar Graph

### 8.3.11 Exercises on Area Bounded by a Polar Graph

- Exercise 1: Area of a Petal of the Graph  $r = \cos 3\theta$
- Exercise 2: Area Enclosed by Cardioid
- Exercise 3: Area Enclosed by a Spiral
- Exercise 5: Area Enclosed by an Inward Spiral

## Document: 8.4 The Soapbox Problem

Movie:  8.4 The Soapbox Problem

### 8.4.1 Introducing The Soapbox Car Problem

### 8.4.2 Preliminary Discussion: Maximizing a Special Kind of Rational Function

### 8.4.3 Finding the Kinetic Energy of a Rolling Wheel

- The Nature of a Wheel in This Section
- Kinetic Energy of a Stationary Spinning Wheel
- The Kinetic Energy of a Rolling Wheel

### 8.4.4 The Dynamics of a Soapbox Car

- Defining the Soapbox Car
- The Equation of Motion of a Soapbox Car
- Choosing the Radius to Maximize the Rolling Speed
- A Final Note: Looking at The Extreme Cases

## Document: 8.5 Conic Curves

**Movie:**  **8.5 Conic Curves**

**8.5.1 Introduction to Conic Curves**

**8.5.2 Rectangular Equations of Conic Curves**

A Rectangular Equation of a Parabola  
A Rectangular Equation of an Ellipse  
A Rectangular Equation of a Hyperbola  
Asymptotes of a Hyperbola

**8.5.3 Exercises on Conic Curves**

Exercise 1: A Parametric Form of the Equation of an Ellipse  
Exercise 2: Adding the Distances from a Point on an Ellipse to the Focal Points  
Exercise 3: A Parametric Form of the Equation of a Hyperbola  
Exercise 4: Parametric Form of a Hyperbola Using Hyperbolic Functions  
Exercise 5: Subtracting the Distances from a Point on an Ellipse to the Focal Points  
Exercise 6: The Reflection Property of a Parabola  
Exercise 7: The Reflection Property of an Ellipse

**8.5.4 Polar Equations of Conic Curves**

The Case  $\varepsilon = 0$   
The Case  $\varepsilon > 0$

**Overview of Chapter 9: Sequences and Series**

**Document: 9.1 Limits of Sequences**

**Movie:**  **9.1 Limits of Sequences**

**9.1.1 Introducing the Concepts**

Sequences and Sequence Notation  
Introducing Limits of Sequences  
Convergent Sequences and Divergent Sequences  
Illustrating Convergent and Divergent Sequences

**9.1.2 Elementary Facts About Limits of Sequences**

Limit of a Constant Sequence  
Relating Limits and Inequalities  
The Sandwich Rule for Sequences  
An Analogue of the Sandwich Rule for Infinite Limits  
The Arithmetical Rules for Limits

**9.1.3 Some Exercises on Limits of Sequences**

Exercise 1: The Limit  $\lim_{n \rightarrow \infty} \frac{(-1)^n}{n}$

Exercise 2: The Limit  $\lim_{n \rightarrow \infty} (-1)^n$  Fails to Exist

Exercise 3: The Limit  $\lim_{n \rightarrow \infty} \sqrt[n]{n}$

Exercise 4: The Limit  $\lim_{n \rightarrow \infty} x^n$  when  $x > 1$

Exercise 5: The Limit  $\lim_{n \rightarrow \infty} x^n$  when  $0 < x < 1$

Exercise 6: The Limit  $\lim_{n \rightarrow \infty} x^n$  when  $-1 < x < 1$

Exercise 7: The Limit  $\lim_{n \rightarrow \infty} \frac{(-1)^n \log n}{n}$

Exercise 8: The Limit The Limit  $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$

Exercise 9: The Limit  $\lim_{n \rightarrow \infty} \frac{2^n}{n!}$

Exercise 10: The Limit  $\lim_{n \rightarrow \infty} \frac{5^n}{n!}$

Exercise 11: The Limit  $\lim_{n \rightarrow \infty} \frac{\log(n!)}{n^2}$

Exercise 12: The Important Limit  $\lim_{n \rightarrow \infty} n \left( 1 - \frac{n^p}{(n+1)^p} \right)$

### 9.1.4 Monotone Sequences

Introduction to Monotone Sequences  
A Condition for an Increasing Sequence to Converge  
A Final Note

## Document: 9.2 An Intuitive Motivation of Infinite Series

Movie:  9.2 An Intuitive Motivation of Infinite Series

### 9.2.1 Our Objective in this Section

### 9.2.2 Some Examples to Illustrate Infinite Series

Example 1: The Sum  $0 + 0 + 0 + 0 + 0 + 0 + \dots$

Example 2: The Sum  $1 + 1 + 1 + 1 + 1 + 1 + \dots$

Example 3: Taking  $a_n = \begin{cases} 1 & \text{if } 1 \leq n \leq 4 \\ 0 & \text{if } n \geq 5 \end{cases}$

Example 4: The Infinitely Repeating Decimal  $0.\overline{1}$

Example 5: The Infinitely Repeating Decimal  $0.\overline{9}$

Example 6: The Infinitely Repeating Decimal  $0.\overline{473}$

Example 7: The Sum  $1 + x + x^2 + x^3 + \dots$  When  $-1 < x < 1$

Example 8: The Sum  $1 - x + x^2 - x^3 + x^4 - \dots$  When  $-1 < x < 1$

Example 9: The Sum  $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$

Example 10: The sum  $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$

Example 11: The sum  $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$

Example 12: The sum  $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots$

Example 13: The Equation  $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$

Example 14: The Equation  $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

Example 15: The Equation  $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

Example 16: Comparing the Series Expansions of exp, cos, and sin

Example 17: The Equation  $x^2 = \frac{\pi^2}{3} - \frac{4 \cos x}{1^2} + \frac{4 \cos 2x}{2^2} - \frac{4 \cos 3x}{3^2} + \dots$

### 9.2.3 Concluding Remarks

## Document: 9.3 Introduction to Infinite Series

Movie:  9.3 Introduction to Infinite Series

### 9.3.1 The Series with $n$ th Term $a_n$

### 9.3.2 Convergence and Divergence of Series

### 9.3.3 Some Examples to Illustrate the Idea of a Series

Example 1: The Series  $\sum 0$

Example 2: The Series  $\sum 1$

Example 3: Taking  $a_n = \begin{cases} 1 & \text{if } 1 \leq n \leq 4 \\ 0 & \text{if } n \geq 5 \end{cases}$

Example 4: The Series  $\sum \frac{1}{n(n+1)}$

Example 5: The Series  $\sum \frac{2}{n(n+1)(n+2)}$  for Each  $n$

Example 6: The Geometric Series  $\sum x^{n-1}$

Example 7: The Series  $\sum \log\left(1 + \frac{1}{n}\right)$

Example 8: The Series  $\sum (-1)^{n-1}$

### 9.3.4 The $n$ th Term Criterion for Divergence

Introduction to the  $n$ th Term Criterion for Divergence

Proof of the  $n$ th Term Criterion for Divergence

### 9.3.5 A Return to the Examples of 9.3.3

Example 1: The Series  $\sum 0$

Example 2: The Series  $\sum 1$

Example 3: Taking  $a_n = \begin{cases} 1 & \text{if } 1 \leq n \leq 4 \\ 0 & \text{if } n \geq 5 \end{cases}$

Example 4: The Series  $\sum \frac{1}{n(n+1)}$

Example 5: The Series  $\sum \frac{2}{n(n+1)(n+2)}$

Example 6: The Geometric Series  $\sum x^{n-1}$

Example 7: The Series  $\sum \log\left(1 + \frac{1}{n}\right)$

Example 8: The Series  $\sum (-1)^{n-1}$

### 9.3.6 Some Applications of the $n$ th Term Criterion for Divergence

A Ratio Criterion for Divergence

Testing the Series  $\sum \frac{n!}{6^n}$

Divergence of the Series  $\sum \frac{(2n)!}{(n!)^2}$

Divergence of The Series  $\sum \frac{(-1)^n 4^n (n!)^2}{(2n)!}$

A Problem that We Cannot Solve Right Now: Test the Series  $\sum \frac{(2n)!}{4^n (n!)^2}$

A Limit Form of the Ratio Criterion for Divergence

Divergence of the Series  $\sum \frac{3^n}{n^{10}}$

Divergence of the Series  $\sum \frac{(3^n)(n!)}{n^n}$

### 9.3.7 A Quick Summary of What We Know at Present

## Document: 9.4 Convergence of Nonnegative Series

### Movie: 9.4 Convergence of Nonnegative Series

#### 9.4.1 Introduction to Nonnegative Series

#### 9.4.2 The Integral Comparison Test

Divergence of the Series  $\sum \frac{1}{n}$

Convergence of the Series  $\sum \frac{1}{n^2}$

The General Form of the Integral Comparison Test

The  $p$ -Series

The  $p$ -Series When  $p > 1$

The  $p$ -Series When  $p < 1$

Conclusion: Convergence Criteria for the  $p$ -Series

A Sharper Form of the  $p$ -Series

The Case  $p = 1$

The Case  $p < 1$

The Case  $p > 1$

#### 9.4.3 Optional: A Sharper Type of Integral Comparison

An Extension of the Integral Comparison Test

Euler's Constant

The Limit  $\lim_{n \rightarrow \infty} \sum_{j=n+1}^{2n} \frac{1}{j}$

Summing the Series  $\sum \frac{(-1)^{n-1}}{n}$

Summing the Series  $\sum \frac{1}{n(2n-1)}$

#### 9.4.4 Comparing Series with One Another

The Comparison Test: Inequality Form

The Comparison Test: Limit Form

#### 9.4.5 Some Exercises on The Comparison Test

Exercise 1: Testing the Series  $\sum \frac{\sin^2 n}{n^2}$

Exercise 2: An Unsuccessful Attempt to Test the Series  $\sum \frac{\sin^2 n}{n}$

Exercise 3: Testing the Series  $\sum \frac{n}{n^4 + 7}$

Exercise 4: Testing the Series  $\sum \frac{n}{n^4 - 7}$

Exercise 5: Testing the Series  $\sum \frac{1}{n^{3/2} + n}$

Exercise 6: Testing the Series  $\sum \frac{1}{n^{3/2} - n}$

Exercise 7: Testing the Series  $\sum \frac{n}{\sqrt{n^4 - n^2 + 2}}$

Exercise 8: Testing the Series  $\sum \frac{\log n}{n^2}$

Exercise 9: Testing the Series  $\sum \frac{n \log n}{\sqrt{n^5 - n^2 + 2}}$

- Exercise 10: Testing the Series  $\sum \frac{1}{n^{1+1/n}}$
- Exercise 11: Testing the Series  $\sum \frac{1}{n^{1+(\log n)/n}}$
- Exercise 12: Testing the Series  $\sum \frac{1}{n^{1+(\log n)^2/n}}$
- Exercise 13: Testing the Series  $\sum \left(\frac{n}{n+1}\right)^n$
- Exercise 14: Testing the Series  $\sum \left(\frac{1}{\log n}\right)^3$
- Exercise 15: Testing the Series  $\sum \left(\frac{1}{\log n}\right)^n$
- Exercise 16: Testing the Series  $\sum \left(\frac{1}{\log n}\right)^{\log n}$
- Exercise 17: Testing the Series  $\sum \left(\frac{1}{\log \log n}\right)^{\log n}$
- Exercise 18: Testing the Series  $\sum \left(\frac{1}{\log n}\right)^{\log \log n}$

#### 9.4.6 The Elementary Ratio Tests

Introducing the Ratio Tests

The Ratio Comparison Test

The d'Alembert Ratio Test, Inequality Form

The d'Alembert Ratio Test, Limit Form, Often Known as "The Ratio Test"

#### 9.4.7 Some Exercises that Rely on d'Alembert's Test (Exercises on "The Ratio Test")

- Exercise 1: Testing the Series  $\sum \frac{n^{1000000}}{2^n}$
- Exercise 2: Testing the Series  $\sum \frac{2^n}{n!}$
- Exercise 3: Testing the Series  $\sum \frac{n!}{n^n}$
- Exercise 4: Testing the Series  $\sum \frac{n^{cn}}{n!}$  Given  $c < 1$
- Exercise 5: Testing the Series  $\sum \frac{(2^n)(n!)}{n^n}$
- Exercise 6: Testing the Series  $\sum \frac{(3^n)(n!)}{n^n}$
- Exercise 7: An Unsuccessful Attempt to Test the Series  $\sum \frac{(e^n)(n!)}{n^n}$
- Exercise 8: An Unsuccessful Attempt to Test the Series  $\sum \frac{n^n}{(e^n)(n!)}$
- Exercise 9: Testing the Series  $\sum \frac{(2n)!}{5^n(n!)^2}$
- Exercise 10: Testing the Series  $\sum \frac{(2n)!}{3^n(n!)^2}$
- Exercise 11: Testing the Series  $\sum \frac{4^n(n!)^2}{(2n)!}$
- Exercise 12: An Unsuccessful Attempt to Test the series  $\sum \frac{(2n)!}{4^n(n!)^2}$
- Exercise 13: Testing the Series  $\sum \frac{((2n)!)^3}{((3n)!)^2}$
- Exercise 14: Testing the Series  $\sum \frac{(\log n)^n}{c^n(\log 2)(\log 3)\cdots(\log n)}$  for  $c > 0$
- Exercise 15: A Second Visit to the Series  $\sum \frac{(e^n)(n!)}{n^n}$

### 9.4.8 The More Powerful Ratio Tests

Introduction to the More Powerful Tests  
The Inequality Form of Raabe's Ratio Test  
The Limit Form of Raabe's Test  
A Level Two Ratio Test  
A Level Three Ratio Test

### 9.4.9 Some Exercises on the More Powerful Ratio Tests

Exercise 1: Successful Testing of the Series  $\sum \frac{(2n)!}{4^n (n!)^2}$

Exercise 2: Testing the Series  $\sum \frac{|\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)|}{n!}$

Exercise 3: Successful Testing of the Series  $\sum \frac{n!}{(e^n)(n!)^2}$

Exercise 4: Testing the Series  $\sum \left( \frac{(2n)!}{4^n (n!)^2} \right)^2$

## Document: 9.5 Absolute and Conditional Convergence

Movie:  9.5 Absolute and Conditional Convergence

### 9.5.1 Introduction to Convergence of Series Whose Terms Can Change Sign

### 9.5.2 Absolutely Convergent Series

Definition of an Absolutely Convergent Series  
Convergence of Absolutely Convergent Series  
Some Examples of Absolutely Convergent Series

### 9.5.3 Conditionally Convergent Series

### 9.5.4 The Alternating Series Test

Statement of the Alternating Series Test  
Warning: Read the Statement of the Alternating Series Test Carefully!  
Some Examples of Series Whose Conditional Convergence Can be Deduced from the Alternating Series Test  
Proof of the Alternating Series Test  
An Error Estimate for Alternating Series  
Approximations to  $\log 2$

### 9.5.5 Dirichlet's Test (Optional)

Statement of Dirichlet's Test  
Proof of Dirichlet's Test  
An Error Estimate for a Series Tested by Dirichlet's Test

### 9.5.6 Some Exercises on Dirichlet's Test (Optional)

Exercise 1: Testing the Series  $\sum \frac{\sin nx}{n}$

Exercise 2: Testing the Series  $\sum \frac{\cos nx}{n}$

Exercise 3: Testing the Series  $\sum \frac{\cos^2 nx}{n}$

Exercise 4: Testing the Series  $\sum \frac{\sin^2 nx}{n}$

Exercise 5: Conditional Convergence of  $\sum \frac{\sin nx}{n}$  and  $\sum \frac{\cos nx}{n}$

Exercise 6: A Relationship Between  $\sum a_n$  and  $\sum a_n^3$

### 9.5.7 Some Further Series to Test with the Alternating Series Test or Dirichlet's Test (Optional)

Introducing This Topic

A Special Technique for Testing Alternating Series

Testing the Series  $\sum \frac{(-1)^n (2n)!}{4^n (n!)^2}$

Testing the Series  $\sum \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!}$

Testing the Series  $\sum \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right)\cdots\left(1 - \frac{1}{n}\right)(-1)^n$

Testing the Series  $\sum \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right)\cdots\left(1 - \frac{1}{n^2}\right)(-1)^n$

## Document: 9.6 Power Series

Movie:  9.6 Power Series

### 9.6.1 Introduction to Power Series

### 9.6.2 Some Examples of Power Series

Example 1: The Geometric Series  $\sum x^n$

Example 2: The Series  $\sum \frac{x^n}{n!}$

Example 3: The Series  $\sum \frac{(-1)^{n-1} x^n}{n}$

Example 4: The Series  $\sum \frac{(-1)^{n-1} (x-4)^{2n-1}}{2n-1}$

Example 5: The Series  $\sum n(x+3)^{n-1}$

Example 6: The Series  $\sum (n!)x^n$

### 9.6.3 Radius and Interval of Convergence of a Power Series

The Case  $0 < r < \infty$

The Case  $r = 0$

The Case  $r = \infty$

### 9.6.4 Some Exercises on Radius and Interval of Convergence

Exercise 1: The Series  $\sum \frac{(x-5)^n}{2^n n^2}$

Exercise 2: The Series  $\sum \frac{(x-5)^n}{2^n n}$

Exercise 3: The Series  $\sum \frac{(-1)^n (x-5)^n}{2^n n}$

Exercise 4: The Series  $\sum \frac{n(x-5)^n}{2^n}$

Exercise 5: The Series  $\sum \frac{(x-5)^{2n}}{n 3^n}$

Exercise 6: The Series  $\sum \frac{(n!)^2}{(2n)!} x^n$

Exercise 7: The Series  $\sum \frac{(2n)!}{(n!)^2} x^n$

Exercise 8: The Binomial Series  $\sum \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!} x^n$

### 9.6.5 The Principal Facts About Power Series

- The Derivative of the Sum of a Power Series
- Higher Derivatives of the Sum of a Power Series
- A Formula for the Coefficients of a Power Series
- The Taylor and Maclaurin Series of a Given Function

### 9.6.6 Some Important Examples of Taylor Series

Example 1: The Geometric Series

Example 2: The Alternating Geometric Series

Example 3: The Equation  $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} = \log(1+x)$ , Found by the Derivative Method

Example 4: The Equation  $\sum_{n=0}^{\infty} \frac{(1)^n}{2n+1} x^{2n+1} = \arctan x$ , Found by the Derivative Method

Example 5: The Equation  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ , Found by the Derivative Method

Example 6: The Equation  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ , Found by the Remainder Method

Example 7: The Equations  $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$  and  $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$  Found by the Derivative Method

Example 8: The Equation  $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$  Found by the Remainder Method

Example 9: The Equation  $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$  Found by the Remainder Method

Example 10: A Bump Function

### 9.6.7 The Binomial Expansion

- An Introduction to the Binomial Series
- The Binomial Coefficients
- A Needed Fact About the Binomial Coefficients
- A Needed Fact About the Sum of the Binomial Series
- Summing the Binomial Series

### 9.6.8 Abel's Theorem

### 9.6.9 Some Applications of Abel's Theorem

Example 1: The Equation  $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = \log 2$

Example 2: The Equation  $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}$

Example 3: The Equation  $\sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{2^{2n} (n!)^2} = \frac{1}{\sqrt{2}}$

### 9.6.10 Tauber's Theorem

**The chapters below are under construction.**

**They may lack polish and their video versions are not yet implemented.**

 **Overview of Chapter 10: Some Basics in Linear Algebra**

 **Document: 10.1 A Glance at Second and Third Order Determinants**

**Movie:**  **10.1 A Quick Look at Second and Third Order Determinants**


 **10.1.1 Second Order Determinants**


Definition of a Second Order Determinant  
Example 1  
Example 2  
Solving Two Equations for Two Unknowns

 **10.1.2 Third Order Determinants**

Definition of a Third Order Determinant  
Alternative Expansions of a Third Order Determinant  
Example of a Third Order Determinant  
Solving Three Equations for Three Unknowns

 **10.1.3 More General Determinants**

 **Document: 10.2 Vectors in Space**

**Movie:**  **10.2 Vectors in Space**

 **10.2.1 Preliminary Note**

 **10.2.2 Introducing the Arithmetical Operations in  $\mathbf{R}^n$**

Definition of the Space  $\mathbf{R}^n$   
Addition and Subtraction in  $\mathbf{R}^n$

 **10.2.3 Some Properties of Addition and Subtraction in  $\mathbf{R}^n$**

Adding any Point to the Origin  
The Commutative Law for Addition in  $\mathbf{R}^n$   
The Associative Law for Addition in  $\mathbf{R}^n$   
Some Facts About Subtraction  
The Symbol  $-A$

 **10.2.4 Scalar Multiplication in  $\mathbf{R}^n$**

Introducing Scalar Multiplication  
Definition of Scalar Multiplication in  $\mathbf{R}^n$   
Some Properties of Scalar Multiplication in  $\mathbf{R}^n$

 **10.2.5 Linear Combinations**

Definition of a Linear Combination  
Two Examples of Linear Combinations  
Example 1

Example 2  
The Standard Basis in  $\mathbf{R}^n$   
The Standard Basis in  $\mathbf{R}^2$   
The Standard Basis in  $\mathbf{R}^3$   
Extending the Idea of Standard Basis to  $\mathbf{R}^n$

### 10.2.6 Geometric Interpretation of the Arithmetical Operations in $\mathbf{R}^2$ and $\mathbf{R}^3$

Norm of a Point in  $\mathbf{R}^2$   
Using the Norm to Find the Length of a Line Segment in  $\mathbf{R}^2$   
Coordinate Axes and the Norm in  $\mathbf{R}^3$   
Using the Norm to Find the Length of a Line Segment in  $\mathbf{R}^3$   
Line Segments with the Same Length and Direction  
The Parallelogram Rule for Addition  
Line Segments with the Same Direction and Different Lengths  
Line Segments with Opposite Directions  
Norm of a Point in  $\mathbf{R}^n$   
Some Simple Facts About the Norm in  $\mathbf{R}^n$   
The Norm is Zero only at  $O$   
The Norm and Scalar Multiplication  
Dividing a Vector by its Norm to Produce a Unit Vector

### 10.2.7 Exercises on the Geometric Interpretation of the Arithmetical Operations in $\mathbf{R}^2$ and $\mathbf{R}^3$

Exercise 1: Midpoint of a Line Segment  
Exercise 2: An Application to Geometry  
Exercise 3: An Application to Geometry  
Exercise 4: An Application to Geometry  
Exercise 5: An Application to Geometry

### 10.2.8 The Concept of a Vector

Motivating the Vector Concept by Looking at Forces that Act on a Particle  
Introducing the Concept of a Vector  
Another Look at Vector Addition

### 10.2.9 The Inner Product (Dot Product)

Preliminary Discussion of the Inner Product (Dot Product)  
Definition of the Inner Product  
The Inner Product of a Point with Itself  
The Commutative Law for the Inner Product  
The Inner Product and Scalar Multiplication  
The Distributive Law for the Inner Product  
Inner Product of Points with Norm One  
The Cauchy-Schwarz Inequality  
The Minkowski Inequality  
The Triangle Inequality  
A Geometric Interpretation of the Inner Product in  $\mathbf{R}^2$  and  $\mathbf{R}^3$   
Perpendicular Line Segments in  $\mathbf{R}^2$  and  $\mathbf{R}^3$   
Orthogonality in  $\mathbf{R}^n$   
Orthonormal Sets  
Expressing Any Vector in Terms of an Orthonormal Set

### 10.2.10 Some Exercises on the Inner Product

Exercise 1  
Exercise 2

- Exercise 3
- Exercise 4: An Application to Geometry
- Exercise 5: An Application to Geometry
- Exercise 6: An Application to Geometry
- Exercise 7: An Application to Geometry

**10.2.11 The Cross Product in  $\mathbf{R}^3$**

- Definition of the Cross Product in  $\mathbf{R}^3$
- Some Examples of Cross Products
  - Example 1
  - Example 2
- The Equation  $\mathbf{A} \times \mathbf{A} = \mathbf{O}$
- The Equation  $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$
- The Distributive Law for the Cross Product
- The Cross Product and Scalar Multiplication
- Failure of the Associative Law
- The Scalar Triple Product
- The Vector Triple Product
- The Norm of a Cross Product
- The Direction of  $\mathbf{A} \times \mathbf{B}$

**10.2.12 Some Exercises on Cross Products**

- Exercise 1: An Application to Area of a Triangle
- Exercise 2: An Application to Area of a Triangle
- Exercise 3: Finding the Area of a Given Triangle
- Exercise 4
- Exercise 5

**10.2.13 Volume of a Parallelepiped**

**Document: 10.3 Lines and Planes in  $\mathbf{R}^3$**

**Movie:**  **10.3 Lines and Planes in  $\mathbf{R}^3$**

**10.3.1 Lines and Parametric Lines in  $\mathbf{R}^2$**

- Introduction to This Section
- Straight Line Graphs of the type  $ax + by = d$  in  $\mathbf{R}^2$
- Parametric Form of the Equation of a Straight Line in  $\mathbf{R}^2$

**10.3.2 Some Exercises on Lines in  $\mathbf{R}^2$**

- Exercise 1: Finding The Intersection of Two Lines
- Exercise 2: Finding The Intersection of Two Parametric Lines
- Exercise 3: A Line Perpendicular to Given Direction
- Exercise 4: Dropping a Perpendicular to a Line
- Exercise 5: Dropping a Perpendicular to a Parametric Line

**10.3.3 Lines and Planes in  $\mathbf{R}^3$**

- The Two Kinds of Equation
- The Equation of a Plane
- Parametric Equations of a Line

**10.3.4 Exercises on Lines and Planes**

- Exercise 1: Equation of a Plane Containing a Given Point and Perpendicular to a Given Direction
- Exercise 2: Equation of a Plane Containing Three Given Points
- Exercise 3: Equation of a Line Containing Two Given Points
- Exercise 4: Equation of a Line Containing a Given Point and Having a Given Direction
- Exercise 5: Intersection of a Line and a Plane
- Exercise 6: Failure of Intersection of a Line and a Plane
- Exercise 7: Intersection of Two Lines
- Exercise 8: Angle Between Two Given Lines
- Exercise 9: Plane Containing Two Given Lines
- Exercise 10: Plane Containing Three Given Points
- Exercise 11: Plane Containing a Line and a Point
- Exercise 12: Line Perpendicular to Two Given Lines
- Exercise 13: Dropping a Perpendicular to a Line
- Exercise 14: Point in a Line Closest to a Given Point
- Exercise 15: Common Perpendicular Between Two Lines
- Exercise 16: Perpendicular from a Point to a Plane


 **10.3.5 Parametric Equation of a Plane in  $R^3$**

- Introducing the Parametric Equation of a Plane
- An Example of a Parametric Equation of a Plane

 **Overview of Chapter 11: Multivariable Differential Calculus**

 **Document: 11.1 Surfaces and Curves in  $R^3$**

**Movie:**  **11.1 Surfaces and Curves in  $R^3$**

 **11.1.1 Preliminary Note on This Section**

 **11.1.2 Surfaces as Implicit Plots and Parametric Surfaces**

 **11.1.3 Some Examples of Surfaces**

- Example 1: Plotting a Cone
- Example 2: Plotting a Circular Paraboloid
- Example 3: Plotting an Ellipsoid
- Example 4: Plotting a Cone and a Hemisphere
- Example 5: Plotting an Hyperboloid of One Sheet
- Example 6: Plotting an Hyperboloid of Two Sheets
- Example 7: Plotting a Corkscrew
- Example 8: Plotting A Double Sea Shell
- Example 9: Plotting a Cylinder
- Example 10: Plotting Möbius Band
- Example 11: Plotting a Cylinder with Two Twists
- Example 12: Plotting a Cylinder with Three Twists
- Example 13: Plotting a Cylinder with Four Twists
- Example 14: Twisting a Cylinder
- Example 15: A Surface with a Surprise

 **11.1.4 Parametric Curves**

- Motivating the Idea of a Parametric Curve in  $R^3$
- 11.1.4.2 Definition of a Parametric Curve in  $R^3$

— **11.1.5 Some Examples of Curves**

- Example 1: Plotting a Spiral on a Cylinder
- Example 2: Plotting a Spiral on a Cone
- Example 3: Plotting an Exponential Spiral
- Example 4: Plotting Two Interlocking Closed Curves
- Example 5: Plotting the Hardy-Walker Knotted Closed Curve

— **Document: 11.2 The Calculus of Curves**

**Movie:**  **11.2 The Calculus of Curves**

— **11.2.1 Limits and Continuity of Parametric Curves**

- Limit of a Parametric Curve at a Given Number
- Continuity of a Curve

— **11.2.2 Some Examples to Illustrate Limits and Continuity of Curves**

- Example 1
- Example 2
- Example 3

— **11.2.3 Velocity, also called the Derivative of a Curve**

- Definition of the Velocity of a Curve
- Speed of a Curve
- Acceleration of a Curve
- An Example to Illustrate the Velocity, Speed and Acceleration of a Curve

— **11.2.4 Geometric Interpretation of Velocity and Speed**

- The Direction of the Velocity of a Curve
- Using Speed to Find the Length of a Curve

— **11.2.5 Some Exercises on Velocity and Speed**

- Exercise 1
- Exercise 2
- Exercise 3
- Exercise 4
- Exercise 5
- Exercise 6
- Exercise 7
- Exercise 8

— **11.2.6 Curvature, Principal Normal, Binormal, and Torsion of a Curve**

- Velocity of a Curve Whose Norm is Constant
- Unit Tangent Vector of a Parametric Curve
- Principal Normal of a Parametric Curve
- The Curvature of a Parametric Curve
- The Equation  $T'(t) = k(t)s'(t)N(t)$
- The Curvature of a Circle is the Reciprocal of Its Radius
- Center of Curvature and Evolute of a Parametric Curve
- The Binormal of a Parametric Curve
- The Orthonormal Triple  $\{T(t), N(t), B(t)\}$
- The Torsion of a Parametric Curve

The Frenet Formulas

— **11.2.7 The Acceleration of a Parametric Curve**

Definition of Acceleration of a Parametric Curve  
The Relationship Between Acceleration, Curvature and Principal Normal  
The Product  $P'(t) \times P''(t)$  and a Useful Formula for  $k(t)$

— **11.2.8 Some Exercises on Curvature**

Exercise 1  
Exercise 2  
Exercise 3  
Exercise 4  
Exercise 5

— **11.2.9 Motion of a Particle in Space: Newton's Law**

The Basic Definitions  
Newton's Law  
Expressing the Force Acting on a Particle in Terms of Curvature

— **11.2.10 Planetary Motion**

Solving the Equation  $f''(x) + f(x) = c$   
The Equation  $f''(x) + f(x) = 0$   
The Equation  $f''(x) + f(x) = c$   
An Alternative Form of the Solution  
Introduction to Planetary Motion  
An Analysis of Planetary Motion

— **Document: 11.3 Real Valued Functions**

**Movie:**  **11.3 Real Valued Functions**

— **11.3.1 Introduction to Real Valued Functions**

— **11.3.2 Some Examples of Real Valued Functions**

Example 1  
Example 2  
Example 3  
Example 4  
Example 5  
Example 6


— **11.3.3 Limits of Real Valued Functions**

Closeness in the Space  $\mathbf{R}^2$   
Closeness in the Space  $\mathbf{R}^3$   
Limit at a Given Point in  $\mathbf{R}^2$   
Limit at a Given Point in  $\mathbf{R}^3$

— **11.3.4 Some Examples of Limits**

Example 1  
Example 2  
Example 3  
Example 4

- Example 5
- Example 6
- Example 7
- Example 8

 **Document: 11.4 Partial Derivatives**

**Movie:**  **11.4 Partial Derivatives**

 **11.4.1 Introduction to Partial Derivatives**

- Partial Derivatives of a Function of Two Variables
- Functions of More than Two Variables
- A Geometric Interpretation of Partial Derivatives
- A More Precise Approach to Partial Derivatives
- Higher Order Partial Derivatives
- Equality of Second Order Mixed Partial Derivatives

 **11.4.2 Some Exercises on Partial Derivatives**


- Exercise 1
- Exercise 2
- Exercise 3
- Exercise 4
- Exercise 5
- Exercise 6
- Exercise 7
- Exercise 8
- Exercise 9

 **11.4.3 The Chain Rule**

- An Example to Motivate the Chain Rule
- A Second Example to Motivate the Chain Rule
- The Chain Rule for Functions of Two Variables
- The Chain Rule for Functions of Three Variables
- The Chain Rule for Functions of  $n$  Variables

 **11.4.4 Some Exercises on the Chain Rule**

- Exercise 1
- Exercise 2
- Exercise 3
- Exercise 4
- Exercise 5
- Exercise 6
- Exercise 7
- Exercise 8: Euler's Formula for Homogeneous Functions

 **Document: 11.5 Vector Fields**

**Movie:**  **11.5 Vector Fields**

 **11.5.1 Introduction to Vector Fields**

- The Force of Gravity as a Vector Field
- Velocity of a Flowing Fluid as a Vector Field

Definition of a Vector Field  
Scalar Fields

**11.5.2 Some Examples of Vector Fields**

Example 1  
Example 2  
Example 3  
Example 4

**11.5.3 Gradient, Divergence, Laplacian, and Curl**

Gradient of a Real Function  
Gradient of a Real Function  
The Laplacian  
The Curl of a Vector Field  
The Operator  $\nabla$  Called Nabla or Del

**11.5.4 Exercises on Gradient, Curl, and Divergence**

Exercise 1  
Exercise 2  
Exercise 3  
Exercise 4  
Exercise 5  
Exercise 6  
Exercise 7  
Exercise 8

**11.5.5 Conservative Vector Fields and Potential of a Field**

Potential of a Vector Field  
A Necessary Condition a Vector Field to be Conservative

**11.5.6 Exercises on Conservative Fields and Potential**

Exercise 1  
Exercise 2  
Exercise 3  
Exercise 4  
Exercise 5

**11.5.7 Directional Derivative**

Motivating the Idea of a Directional Derivative  
Definition of the Directional Derivative of a Scalar Field  
A Useful Formula for a Directional Derivative

**11.5.8 Exercises on Directional Derivatives**

Exercise 1  
Exercise 2  
Exercise 3

**Document: 11.6 Further Topics on Partial Differentiation**

**Movie:**



**11.6 Further Topics on Partial Differentiation**

**11.6.1 A Quick Look at Matrix Arithmetic**

Notation for Matrices  
 Addition and Subtraction of Matrices  
 Multiplication of a Matrix by a Number  
 Multiplication of One Matrix by Another  
 The Identity Matrix  
 Invertible and Singular Matrices  
 A Relationship Between Matrix Multiplication and Determinants

**11.6.2 Some Exercises on Matrix Arithmetic**

Exercise 1  
 Exercise 2  
 Exercise 3  
 Exercise 4  
 Exercise 5

**11.6.3 The Jacobian Matrix of a Vector Field**

Writing the Coordinates of a Vector Field Vertically  
 Motivating the Idea of a Jacobian Matrix  
 The Jacobian Matrix of a Vector Field in  $\mathbf{R}^3$   
 The Jacobian Matrix of a Function from a Region in  $\mathbf{R}^6$  into  $\mathbf{R}^4$   
 The General Case of a Jacobian Matrix

**11.6.4 Expressing the Chain Rule in Matrix Form**

A Simple Example Showing the Chain Rule in Matrix Form  
 Revisiting the Chain Rule for Real Functions  
 The  $4 \times 2 \times 3$  Form of the Chain Rule  
 The General  $n \times m \times k$  Form of the Chain Rule

**11.6.5 Implicit Differentiation**

A Review of Implicit Differentiation as We Saw It in Section 3.8  
 Applying Implicit Differentiation to a Single Equation in Three Unknowns  
 Applying Implicit Differentiation to Two Equations in Three Unknowns: A Special Case  
 Applying Implicit Differentiation to Two Equations in Three Unknowns: The General Case  
 Applying Implicit Differentiation to Four Equations in Seven Unknowns:  
 The General Implicit Differentiation Problem

**11.6.6 Principal Normal of a Parametric Surface**

Introducing the Concept of Principal Normal  
 Principal Normal of a Sphere  
 Principal Normal of a Cone  
 Finding a Normal to a Surface of the Form  $f(x, y, z) = 0$   
 Tangent Plane to the Surface  $x^2y + yz^2 = 20$  at  $(1, 2, 3)$   
 Tangent Plane to the Surface  $x^3 + y^3 + z^3 + 3xyz = 6$  at  $(1, 1, 1)$   
 Tangent Plane to the Surface  $ze^{xy} - 4x^2 - 4y^2 = e - 8$  at  $(1, 1, 1)$   
 Tangent Plane to the Surface  $e^{-x^2-y^2-z^2}(4x^2 + 5xyz + 4y^2 + 4z^2) = 17e^{-3}$  at  $(1, 1, 1)$

**Document: 11.7 Maxima and Minima**

**Movie:**  **11.7 Maxima and Minima**

**11.7.1 Definitions of Maxima and Minima**

Definition of Maximum and Minimum of a Function

Definition of Local Maximum and Local Minimum of a Function

**11.7.2 Some Examples to Illustrate the Definitions**

- Example 1
- Example 2
- Example 3
- Example 4

**11.7.3 Basic Facts About Maxima and Minima**

- Existence of Maxima and Minima of a Function
- Fermat's Theorem
- Critical Points of a Function
- Finding Maxima and Minima of a Given Function
- Saddle Points
- The Second Derivative Test for Maxima and Minima

**11.7.4 Exercises on Maxima and Minima**

- Exercise 1
- Exercise 2
- Exercise 3
- Exercise 4
- Exercise 5: A Box Problem
- Exercise 6: A Maximum Minimum Problem that Requires a Computer Algebra System

**11.7.5 The Standard Simplex in  $\mathbf{R}^n$**

- The Standard Simplex in  $\mathbf{R}^1$ ,  $\mathbf{R}^2$ , and  $\mathbf{R}^3$
- Definition of the Standard Simplex  $Q^n$
- A Maximum Minimum Problem on the Simplex  $Q^n$
- An Application of the Preceding Maximum Minimum Problem

**Overview of Chapter 12: Multivariable Integral Calculus**

**Document: 12.1 Integration on Curves**

**Movie:**  **12.1 Integration on Curves**

**12.1.1 Integration on a Smooth Curve**

- Definition of a Smooth Curve
- Integrals of the Type  $\int_p f dx$ ,  $\int_p f dy$ , and  $\int_p f dz$
- Integrals of the Type  $\int_p F \cdot dP = \int_p F \cdot (dx, dy, dz) = \int_p f dx + g dy + h dz$
- Application to Work Done by a Force

**12.1.2 Examples of Integrals on Smooth Curves**

- Example 1
- Example 2
- Example 3

**12.1.3 Fundamental Theorem of Calculus for Integrals on Curves**

- Introduction to the Fundamental Theorem
- Statement of the Fundamental Theorem for Integrals of the Type  $\int_p F \cdot dP$

Path Independence and the Fundamental Theorem  
The Role of “Whirlpools”

 **12.1.4 Exercises on Integrals on Curves**

- Exercise 1
- Exercise 2
- Exercise 3
- Exercise 4
- Exercise 5
- Exercise 6

 **12.1.5 Reparametrizing a Curve**

- Motivating the Idea of a Reparametrization of a Curve
- Reparametrizing a Curve in the Direction of Travel
- Reparametrizing a Curve Reversing the Direction of Travel
- An Animation to Illustrate a Reparametrization that Reverses the Direction of Travel
- Integrating on a Reparametrization that is in the Direction of Travel
- Integrating on a Reparametrization that Reverses the Direction of Travel

 **12.1.6 Integration on a Chain of Smooth Curves**

- Motivating the Idea of a Chain of Curves
- Definition of a Chain of Curves
- Integrating on a Chain of Curves
- Integrating around a Triangle

 **12.1.7 Exercises on Integrals on Chains**

- Exercise 1
- Exercise 2
- Exercise 3

 **12.1.8 A More General Notion of a Chain of Curves**

 **Document: 12.2 Integration of a Function of Two Variables**

**Movie:**  **12.2 Integration of a Function of Two Variables**

 **12.2.1 Iterated Integrals in Two Variables**

- Iterated Integrals with Constant Limits
- More General Iterated Integrals

 **12.2.2 Some Examples of Iterated Integrals**

- Example 1
- Example 2
- Example 3
- Example 4
- Example 5
- Example 6
- Example 7
- Example 8

 **12.2.3 The Fichtenholz Theorem**

- Note to Instructors on the Fichtenholz Theorem

Introduction to Fichtenholz Theorem  
Statement of the Fichtenholz Theorem

— **12.2.4 Some Exercises on Iterated Integrals**

Exercise 1  
Exercise 2  
Exercise 3  
Exercise 4  
Exercise 5  
Exercise 6  
Exercise 7  
Exercise 8

— **12.2.5 Introduction to Integration over Regions**

— **12.2.6 Integrals over Regions in  $\mathbf{R}^1$**

Integral over an Interval  $[a,b]$  in  $\mathbf{R}^1$   
The General Case of a Region in  $\mathbf{R}^1$

— **12.2.7 Some Examples to Illustrate the Definition of  $\int_S f(x)dx$**

Example 1  
Example 2  
Example 3  
Example 4

— **12.2.8 Integrals over Regions in  $\mathbf{R}^2$**

— **12.2.9 Exercises on Double Integrals**

Exercise 1  
Exercise 2  
Exercise 3  
Exercise 4  
Exercise 5  
Exercise 6  
Exercise 7  
Exercise 8  
Exercise 9  
Exercise 10  
Exercise 11  
Exercise 12  
Exercise 13

— **12.2.10 Approximating Double Integrals by Sums**

Darboux's Theorem  
Using A Double Integral to Find Area  
Using a Double Integral to Find the Value of a Metal Plate  
Using a Double Integral to Find Volume

— **12.2.11 Exercises on Applications of Double Integrals**

Exercise 1  
Exercise 2  
Exercise 3  
Exercise 4

- Exercise 5: The Plumber's Nightmare
- Exercise 6
- Exercise 7
- Exercise 8

**Document: 12.3 The Gamma and Beta Functions**

**Movie:  12.3 The Gamma and Beta Functions**

**12.3.1 The Equation  $\lim_{x \rightarrow \infty} \frac{x^p}{e^x} = 0$**

The Equation  $\lim_{x \rightarrow \infty} \frac{x^0}{e^x} = 0$

The Equation  $\lim_{x \rightarrow \infty} \frac{x^p}{e^x} = 0$  When  $p$  Is Negative

The Equation  $\lim_{x \rightarrow \infty} \frac{x^p}{e^x} = 0$  When  $p$  Is Positive

**12.3.2 Introducing the Gamma Function**

- Definition of the Gamma Function
- Some Examples to Illustrate the Gamma Function
- A Harder Example
- The Convergence of the Integral  $\int_0^{\infty} x^{a-1} e^{-x} dx$
- The Graph of the Gamma Function

**12.3.3 Some Elementary Facts About the Gamma Function**

- The Recurrence Formula
- The Gamma Function and Factorials
- The Substitution  $x = t^2$
- The Value of  $\Gamma\left(\frac{1}{2}\right)$

**12.3.4 Introducing the Beta Function**

- Definition of the Beta Function
- Some Examples to Illustrate the Beta Function
- The Convergence of the Integral  $\int_0^1 t^{a-1} (1-t)^{b-1} dt$
- The Graph of the Beta Function

**12.3.5 Some Elementary Facts About the Beta Function**

- Symmetry of the Beta Function
- The Substitution  $u = ct$
- The Substitution  $t = \sin^2 \theta$
- The Value of  $B\left(\frac{1}{2}, \frac{1}{2}\right)$

**12.3.6 The Relationship Between the Gamma and Beta Functions**

- Introducing the Relationship
- Proof of the Formula  $\Gamma(a)\Gamma(b) = \Gamma(a+b)B(a,b)$

**12.3.7 Some Exercises on the Gamma and Beta Functions**

- Exercise 1
- Exercise 2
- Exercise 3
- Exercise 4
- Exercise 5


- Exercise 6
- Exercise 7
- Exercise 8
- Exercise 9
- Exercise 10
- Exercise 11
- Exercise 12
- Exercise 13
- Exercise 14

 **12.3.8 A Hard Fact About the Gamma Function**


- Statement of the Hard Fact
- An Application of the Hard Fact

 **Document: 12.4 Changing Integrals to Polar Coordinates**

**Movie:**  **12.4 Changing Integrals to Polar Coordinates**

 **12.4.1 Introducing the Change to Polar Coordinates**

- A First Look at the Method
- A More Careful Description of the Regions of Integration
- Motivating the Formula for Changing to Polar Coordinates

 **12.4.2 Exercises on Polar Coordinates**

- Exercise 1
- Exercise 2
- Exercise 3
- Exercise 4
- Exercise 5
- Exercise 6
- Exercise 7
- Exercise 8
- Exercise 9
- Exercise 10
- Exercise 11
- Exercise 12
- Exercise 13
- Exercise 14

 **Document: 12.5 Integration of a Function of Three Variables**

**Movie:**  **12.5 Integration of a Function of Three Variables**

 **12.5.1 Iterated Integrals in Three Variables**

- Iterated Integrals with Constant Limits
- More General Iterated Integrals

 **12.5.2 Some Examples of Iterated Integrals in Three Variables**

- Example 1
- Example 2
- Example 3
- Example 4

Example 5

— **12.5.3 The Fichtenholz Theorem**

— **12.5.4 Integration over Regions in  $\mathbf{R}^3$**

Definition of the Integral over a Region in  $\mathbf{R}^3$   
Darboux's Theorem  
Using a Triple Integral to Find Volume  
Using a Triple Integral to Find the Mass of a Region  
Using a Triple Integral to Find the Value of a Metal Solid

— **12.5.5 Some Exercises on the Conversion of Triple Integrals to Iterated Integrals**

Exercise 1  
Exercise 2  
Exercise 3  
Exercise 4  
Exercise 5

— **12.5.6 Cylindrical Coordinates**

Introduction to Cylindrical Coordinates  
Cylindrical Coordinates with  $\theta$  Changing  
Cylindrical Coordinates with  $r$  Changing  
Cylindrical Coordinates with  $z$  Changing

— **12.5.7 Exercises on Cylindrical Coordinates**

Exercise 1  
Exercise 2  
Exercise 3

— **12.5.8 Spherical Coordinates**

Introduction to Spherical Coordinates  
Spherical Coordinates with  $\theta$  Changing  
Spherical Coordinates with  $\rho$  Changing  
Spherical Coordinates with  $\varphi$  Changing

— **12.5.9 Changing Integrals to Spherical Coordinates**

A First Look at the Method  
A More Careful Description of the Regions of Integration  
Motivating the Formula for Changing to Spherical Coordinates

— **12.5.10 Exercises on Spherical Coordinates**

Exercise 1  
Exercise 2  
Exercise 2  
Exercise 4  
Exercise 5  
Exercise 6  
Exercise 7  
Exercise 8  
Exercise 9  
Exercise 10: Finding the Centroid of a Solid Region  
Exercise 11: Finding the Moment of Inertia of a Solid Region

 **Document: 12.6 Changing Variable in a Multiple Integral**

**Movie:**  **12.6 Changing Variable in a Multiple Integral**

 **12.6.1 The Change of Variable Theorem for Integrals of Functions of a Single Variable**

Two Versions of the Change of Variable Formula

Review of the Change of Variable Formula for Integrals Between Limits

Some Notes About the Change of Variable Formula for Integrals Between Limits

The Function  $u$  May Be Increasing or Decreasing or Neither Increasing nor Decreasing

As  $x$  Runs from  $a$  to  $b$ , There Is No Reason to Expect that  $u(x)$  Stays Between  $u(a)$  and  $u(b)$

The Quantity  $u(x)$  Can Run Several Times Between  $u(a)$  and  $u(b)$

The Change of Variable Formula for Integration on Intervals

When the Function  $u$  Is Increasing

When the Function  $u$  Is Decreasing

Combining the Two Cases

What Happens if  $u$  is Neither Increasing nor Decreasing?

 **12.6.2 The Change of Variable Theorem for Double Integrals**

Introduction to the Change of Variable Formula for Two Variables

Revisiting the Change to Polar Coordinates to Illustrate the Change of Variable Formula

Motivating the Change of Variable Formula

 **12.6.3 Exercises on Change of Variable for Double Integrals**

Exercise 1

Exercise 2

Exercise 3

Exercise 4

Exercise 5

Exercise 6

 **12.6.4 The Change of Variable Theorem for Triple Integrals**

Introduction to the Change of Variable Formula for Three Variables

Motivating the Change of Variable Formula

 **12.6.5 Exercises on Change of Variable for Triple Integrals**

Exercise 1

Exercise 2

Exercise 3

 **Document: 12.7 Integrals on Parametric Surfaces**

**Movie:**  **12.7 Integrals on Parametric Surfaces**

 **12.7.1 Preliminary Statement**

 **12.7.2 A Quick Review of Curves and Surfaces**

A Quick Review of Parametric Curves

A Quick Review of Parametric Surfaces in  $\mathbf{R}^2$  or  $\mathbf{R}^3$

 **12.7.3 The Boundary of a Parametric Surface**

The Notation  $[A, B]$  if  $A$  and  $B$  are Points in Space  
The Boundary of the Standard 2-Simplex  $Q^2$   
The Boundary of a Rectangle in  $R^2$   
The Boundary of a Parametric Surface in  $R^2$  or  $R^3$   
    When the Domain Region is  $Q^2$   
    When the Domain Region is a Rectangle  
A Formula for Integrating on the Boundary of a Surface  
Simple Closed Curves and Jordan Regions

#### 12.7.4 Green's Theorem

Introduction to Green's Theorem  
Green's Theorem on the Standard 2-Simplex  $Q^2$   
Green's Theorem on a Rectangle  
The General Form of Green's Theorem

#### 12.7.5 Some Exercises on Green's Theorem

Exercise 1: Finishing the Proof of Green's Theorem on a Rectangle  
Exercise 2: Using Green's Theorem to Find Area  
Exercise 3: Finding the Area of a Region  
Exercise 4: Finding the Area of a Region  
Exercise 5: Finding the Area of a Region  
Exercise 6: Using Green's Theorem to Find a Centroid  
Exercise 7: Finding the Centroid of a Region

#### 12.7.6 Integrating on Parametric Surfaces

Integrating on a Parametric Surface in  $R^2$   
Integrating on a Parametric Surface in  $R^2$   
Integrating a Vector Field on a Surface  
A Surface Integral Form of Green's Theorem

#### 12.7.7 Stokes' Theorem

Introduction to Stokes' Theorem  
Statement of Stokes' Theorem  
Proof of Stokes' Theorem

#### 12.7.8 Some Examples to Illustrate Stokes' Theorem

12.7.8.1 Example 1: Stokes' Theorem on a Triangle  
Example 2: Stokes' Theorem on a Portion of Paraboloid  
Example 3: Stokes' Theorem on a Möbius Band  
Example 4: Stokes' Theorem on a Slipped Möbius Band